Construction of anti-symplectic Involutions by categorical method

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1 Preliminaries

1.1 Moduli space of semi-stable sheaves $M_H(v)$ and related results

First we review the construction of $M_H(v)$ on K3 surfaces and cubic 4 folds and some basic results.

Let (X, H) be a polarized K3 surface of genus g, such that $H^2 = 2g - 2$. The Mukai lattice is defined as the integral cohomology:

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

This lattice is endowed with the Mukai pairing. For any object $\mathcal{E} \in D^b(X)$, we define its Mukai vector $v(\mathcal{E}) = (v_0, c_1, v_2) \in H^*(X, \mathbb{Z})$ where

$$v_0 = \mathrm{rk}(\mathcal{E}) \quad c_1 = c_1(\mathcal{E}) \quad v_2 = \chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E})$$

By Riemann-Roch theorem, the last component can be rewritten as:

$$v_2 = \mathrm{rk}(\mathcal{E}) + \frac{c_1(\mathcal{E})^2}{2} - c_2(\mathcal{E})$$

The moduli space $M_H(v)$ consists of H-semistable (torsion-free) sheaves on X with a fixed Mukai vector v.

- *Dimension:* The dimension of $M_H(v)$ is given by the Mukai square plus 2:

$$\dim M_H(v) = v^2 + 2 = c_1^2 - 2v_0v_2 + 2$$

- *Geometric Type:* For specific restrictions on v, $M_H(v)$ is an irreducible holomorphic symplectic (IHS) manifold of dimension $2n$, deformation equivalent to the Hilbert scheme $X^{[n]}$ of points on a K3 surface.

Remark 1.1 (IHS Manifold)**.** A compact Kähler manifold X is *irreducible holomorphic symplectic* (IHS) if it is simply connected and $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$, where σ is a unique, everywhere non-degenerate holomorphic 2-form.

Now we move onto the result of Chunyi Li [LPZ20] for cubic 4 folds, which states the geometry of moduli space for certain Mukai vector v. Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. The bounded derived category $D^b(Y)$ admits the following semiorthogonal decomposition:

$$D^b(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(H), \mathcal{O}_Y(2H) \rangle$$

where H is the hyperplane class. The *Kuznetsov component* $\mathrm{Ku}(Y)$ is a K3 category which means its Serre functor satisfies $S_{\mathrm{Ku}(Y)} \cong [2]$.

The numerical Grothendieck group $K_{num}(\mathrm{Ku}(Y))$ is a lattice of rank at least 2, containing an A_2 sub-lattice spanned by two specific Mukai vectors:

- $\lambda_1 = [\mathrm{pr}_{\mathrm{Ku}(Y)}(\mathcal{O}_L(H))]$
- $\lambda_2 = [\mathrm{pr}_{\mathrm{Ku}(Y)}(\mathcal{O}_L(2H))]$

where $L \subset Y$ is a line and the projection $\mathrm{pr} : D^b(Y) \to Ku(Y)$ equals the composition of left mutations $\mathrm{pr}_{\mathrm{Ku}(Y)}(E) = \mathbb{L}_{\mathcal{O}_Y}\mathbb{L}_{\mathcal{O}_Y(H)}\mathbb{L}_{\mathcal{O}_Y(2H)}(E)$ and $\mathbb{L}_E(F)$ is defined with the aid of the following distinguished triangle:

$$\mathbb{L}_E F \longrightarrow \mathrm{Hom}^\bullet(E,F) \otimes E \longrightarrow F \longrightarrow (\mathbb{L}_E F)[1]$$

The Mukai pairing on $Ku(Y)$ is the Euler form $\langle \lambda_i, \lambda_j \rangle = \chi(\mathcal{O}_L(iH), \mathrm{pr}_{\mathrm{Ku}(Y)}(\mathcal{O}_L(jH)))$ via adjunction.
Computing these characteristics using the Koszul resolution of L gives $\langle \lambda_i, \lambda_i \rangle = -2$ and $\langle \lambda_1, \lambda_2 \rangle = 1$, which means Gram matrix $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

3. Main Isomorphism Theorems

The main results of [LPZ20, Theorem 1.1, 1.2] identify classical hyperkähler manifolds as moduli spaces of Bridgeland stable objects in $Ku(Y)$:

Theorem 1.2 (Fano Variety of Lines)**.** *For any smooth cubic fourfold Y, then for all Bridgeland stability conditions σ on $\mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$, the moduli space of σ-stable objects in $Ku(Y)$ with Mukai vector $\lambda_1 + \lambda_2$ is isomorphic to the Fano variety of lines F_Y:*

$$M_\sigma(\lambda_1 + \lambda_2) \cong F_Y$$

This provides a categorical explanation for the holomorphic symplectic structure on F_Y.

Theorem 1.3 (LLSvS Eightfold)**.** *Let Y be a smooth cubic fourfold not containing a plane, there exist Bridgeland stability conditions in a geometric chamber $C_{geo} \subset \mathrm{Stab}^\dagger(\mathrm{Ku}(Y))$ such that the moduli space of σ-stable objects in $Ku(Y)$ with Mukai vector $2\lambda_1 + \lambda_2$ is isomorphic to the Lehn-Lehn-Sorger-van Straten (LLSvS) eightfold M_Y:*

$$M_\sigma(2\lambda_1 + \lambda_2) \cong M_Y$$

where M_Y is a compact hyperkähler manifold of dimension 8, deformation equivalent to $S^{[4]}$.

Suppose the quadric surface fibration is defined by a quadratic form $q : S^2\mathcal{E} \to \mathcal{L}$ on a rank 4 vector bundle $\mathcal{E}$ over $\mathbb{P}^2$, taking values in a line bundle $\mathcal{L}$.

- *The Even Clifford Algebra ($\mathcal{B}_0$):* The even Clifford algebra $\mathcal{B}_0$ is defined as the quotient of the tensor algebra of $\mathcal{E}^{\otimes 2} \otimes \mathcal{L}^{-1}$ below:

  $$\mathcal{B}_0 = \frac{\bigoplus_{k \geq 0}(\mathcal{E}^{\otimes 2k} \otimes \mathcal{L}^{-k})}{\langle v \otimes v \otimes l^{-1} - q(v)l^{-1} \rangle}$$

  where v is a local section of $\mathcal{E}$ and l is a local section of $\mathcal{L}$. It is a coherent sheaf of $\mathcal{O}_{\mathbb{P}^2}$-algebras of rank $2^{4-1} = 8$. This $\mathcal{B}_0$ uniquely determines the Kuznetsov component via $Ku(Y) \cong D^b(\mathbb{P}^2, \mathcal{B}_0)$ [Kuz08].

- *The Odd Clifford Bimodule ($\mathcal{B}_1$):* The odd part is defined as $\mathcal{B}_1 = \mathcal{B}_0 \otimes \mathcal{E}$. It naturally inherits a left $\mathcal{B}_0$-module structure from the multiplication in $\mathcal{B}_0$. The right $\mathcal{B}_0$-module structure is induced by the fundamental Clifford multiplication rule.

The main structural result for $Ku(Y)$ in this case is:

- *Categorical Equivalence:* $Ku(Y) \cong D^b(\mathbb{P}^2, \mathcal{B}_0)$, the derived category of coherent $\mathcal{B}_0$-modules on $\mathbb{P}^2$ [Kuz08, Theorem 4.3].
- *Relation to K3 Surfaces:* The discriminant of the fibration is a sextic curve $D \subset \mathbb{P}^2$. The double cover $f : X \to \mathbb{P}^2$ branched along D is a K3 surface.

1.2 The Mukai lattice and (anti)-auto-equivalence on $D^b(X)$

Let us briefly review some basic (anti)-auto-equivalences of $D^b(X)$ and review their action on the cohomology of X, here we only need X is a smooth projective variety.

Definition 1.4 (Basic Auto-equivalences)**.**

1. *Shifts* The exact autoequivalence given by the shift functor $E \mapsto E[1]$.
2. *Twisting by Line Bundles:* For any $\mathcal{L} \in \mathrm{Pic}(X)$, the exact autoequivalence given by the tensor product $E \mapsto E \otimes \mathcal{L}$.
3. *Automorphisms of the Variety:* For any $f \in \mathrm{Aut}(X)$, the exact autoequivalence given by the pull-back $E \mapsto f^*E$.
4. *Derived Duality (Anti-equivalence):* The contravariant functor $E \mapsto R\mathcal{H}om(E, \mathcal{O}_X)$.
5. *Spherical Twists:* For a spherical object $\mathcal{S}$, the left mutation functor $T_{\mathcal{S}}(E) = \mathrm{Cone}(R\,\mathrm{Hom}(\mathcal{S}, E) \otimes \mathcal{S} \to E)$.

In fact, these operations naturally induce transformations on the cohomology via their action on the Mukai vector $v(E)$ or Chern character $\mathrm{ch}(E)$.

Proposition 1.5 (Action on Cohomology)**.** *The basic (anti)-autoequivalences on $D^b(X)$ induce the following actions on the cohomology ring $H^*(X, \mathbb{Q})$ via the Mukai vector $v(E) = \mathrm{ch}(E)\sqrt{td(X)}$ or Chern character $\mathrm{ch}(E)$:*

1. *Shifts: $v(E[1]) = -v(E)$.*
2. *Twisting by Line Bundles: $v(E \otimes \mathcal{L}) = v(E) \cdot \mathrm{ch}(\mathcal{L})$.*
3. *Automorphisms: $v(f^*E) = f^*v(E)$.*
4. *Derived Duality: $\mathrm{ch}(R\mathcal{H}om(E, \mathcal{O}_X)) = \mathrm{ch}(E)^\vee$, where the involution $(-)^\vee$ multiplies the degree $2k$ component of $H^*(X)$ by $(-1)^k$.*
5. *Spherical Twists: $v(T_{\mathcal{S}}(E)) = v(E) - \chi(\mathcal{S}, E)v(\mathcal{S})$, where $\chi(\mathcal{S}, E) = \sum(-1)^i \dim \mathrm{Ext}^i(\mathcal{S}, E)$ is the Euler form.*

Remark 1.6. Now we clarify how spherical twists act on $H^*(X, \mathbb{Z})$ of a K3 surface X.

On a K3 surface, the Mukai vector for an object E is defined as $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} = (r, c_1, \mathrm{ch}_2 + r)$.

For a spherical object $\mathcal{S}$ on X, the Euler characteristic is $\chi(\mathcal{S}, \mathcal{S}) = 2$. Under the Mukai pairing $\langle -, - \rangle$, this implies $v(\mathcal{S})^2 = -\chi(\mathcal{S}, \mathcal{S}) = -2$. The spherical twist $T_{\mathcal{S}}(E)$ is defined by the exact triangle:

$$R\,\mathrm{Hom}(\mathcal{S}, E) \otimes \mathcal{S} \longrightarrow E \longrightarrow T_{\mathcal{S}}(E)$$

Because the Mukai vector is additive on exact triangles, we have:

$$v(T_{\mathcal{S}}(E)) = v(E) - \chi(\mathcal{S}, E)v(\mathcal{S})$$

By the Hirzebruch-Riemann-Roch theorem on K3 surfaces, $\chi(\mathcal{S}, E) = -\langle v(\mathcal{S}), v(E)\rangle$, this means $v(T_{\mathcal{S}}(E)) = v(E) + \langle v(\mathcal{S}), v(E)\rangle v(\mathcal{S})$. Since $v(\mathcal{S})$ is a (-2)-class, this coincide with

$$R_{\mathcal{S}}(x) = x - 2\frac{\langle x, v(\mathcal{S})\rangle}{\langle v(\mathcal{S}), v(\mathcal{S})\rangle} v(\mathcal{S})$$

so we obtain the formula

$$v(T_{\mathcal{S}}(E)) = R_{\mathcal{S}}(v(E))$$

A fundamental question is whether the operations above are enough to generate the entire group $\mathrm{Aut}(D^b(X))$. Bondal and Orlov proved that for varieties with ample or anti-ample canonical sheaves, the answer is “yes".

Theorem 1.7 (Bondal-Orlov, 1997)**.** *Let X be a smooth irreducible projective variety with ample either canonical (ω_X) or anticanonical (ω_X^{-1}) sheaf. Then $\mathrm{Aut}(D^b(X))$ is generated strictly by the automorphisms of the variety, the twists by invertible sheaves, and the translations.*

Algebraically, this gives a semi-direct product structure:

$$\mathrm{Aut}(D^b(X)) \cong \mathrm{Aut}(X) \ltimes (\mathrm{Pic}(X) \oplus \mathbb{Z})$$

A natural question is: where are spherical objects in $D^b(X)$ when X is a $K3$-surface? The answer is there are no such objects, here we need to recall the definition of spherical objects.

Definition 1.8 (Spherical Object)**.** Let $\mathcal{D}$ be a k-linear triangulated category with a Serre functor S. An object $E \in \mathcal{D}$ is called *n-spherical* if:

1. $\mathrm{Ext}^i(E, E) \cong \mathbb{C}$ for $i = 0, n$, and 0 otherwise.
2. $S(E) \cong E[n]$.

Remark 1.9. In the setting of [BO97, Theorem 3.1], by definition, the Serre functor is $S(E) \cong E \otimes \omega_X[n]$. The condition $S(E) \cong E[n]$ thus forces $E \otimes \omega_X \cong E$ since the Serre functor is $S = - \otimes \omega_X[n]$.

Because ω_X is ample (or anti-ample), this isomorphism implies that the support of E must be 0-dimensional. Following [BO97, Proposition 2.2], such an object E is isomorphic (up to shift) to a skyscraper sheaf $\mathcal{O}_x$. However, for $\mathcal{O}_x$, we have $\dim \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) = n$, this makes a contradiction with the definition of spherical object.

Therefore, no spherical objects exist in $D^b(X)$ for *Fano* or *General type* varieties of dimension ≥ 2.

2 The Anti-Auto-Equivalence Functor

2.1 Construction of $\Phi^p_{\mathcal{S},\mathcal{L}}$ via spherical twists

Definition 2.1. Let X be a smooth projective variety. Let $\mathcal{S} \in D^b(X)$ be a spherical object and $\mathcal{L} \in \mathrm{Pic}(X)$. We define the contravariant endofunctor $\Phi^p_{\mathcal{S},\mathcal{L}} : D^b(X) \to D^b(X)$ as:

$$\Phi^p_{\mathcal{S},\mathcal{L}}(E) := R\mathcal{H}om(T_{\mathcal{S}}(E), \mathcal{L}[p])$$

where $T_{\mathcal{S}}$ is the spherical twist along $\mathcal{S}$. In the following, we usually fix $p = 1$ and $\mathcal{L} = \mathcal{O}_X(dH)$, denoting the functor as $\Phi_{\mathcal{S},d}$.

A natural question is when does $\Phi^p_{\mathcal{S},\mathcal{L}}$ is an involution, the following lemma answers this question.

Lemma 2.2. *If $\mathcal{S} \simeq R\mathcal{H}om(\mathcal{S}, \mathcal{L}[q])$ for some $q \in \mathbb{Z}$, then the functor $\Phi = \Phi^p_{\mathcal{S},\mathcal{L}}$ is involutive for all $p \in \mathbb{Z}$ (i.e., $\Phi \circ \Phi \simeq \mathrm{id}$). Specifically, if $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot H$ and $\mathcal{S}$ is a torsion-free sheaf of rank s_0, this condition holds if and only if:*

1. $s_0 \in \{1, 2\}$, *and*
2. $\mathcal{L} \simeq \det(\mathcal{S})^{\frac{2}{s_0}}$.

Here we state a result that whether $\Phi^p_{\mathcal{S},\mathcal{L}}$ sends semistable sheaf $\mathcal{E}$ to semistable sheaf, and $\Phi^p_{\mathcal{S},\mathcal{L}}(\mathcal{E})$ not even coherent sheaf, the following result confirms $\Phi^p_{\mathcal{S},\mathcal{L}}$ maps coherent sheaves to coherent sheaves.

Proof. Recall the dual spherical twist $T'_{\mathcal{S}}$ defined as

$$\mathcal{E} \to \mathrm{Cone}(\mathcal{E} \to \mathrm{RHom}(\mathcal{E}, \mathcal{S})^\vee \otimes \mathcal{S})$$

basic property shows $T_{\mathcal{S}} \circ T'_{\mathcal{S}} \simeq T'_{\mathcal{S}} \circ T_{\mathcal{S}} \simeq \mathrm{id}$.

By definition, $\Phi(\mathcal{E}) = R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{E}), \mathcal{L}[1])$. Applying Φ again yields:

$$\Phi(\Phi(\mathcal{E})) = R\mathcal{H}om(T_{\mathcal{S}}(R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{E}), \mathcal{L}[1])), \mathcal{L}[1])$$

Using the given natural isomorphism $T'_{\mathcal{S}}(R\mathcal{H}om(\mathcal{F}, \mathcal{L})) \simeq R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{F}), \mathcal{L})$ with $\mathcal{F} = T_{\mathcal{S}}(\mathcal{E})$, and the assumption $\mathcal{S} \simeq R\mathcal{H}om(\mathcal{S}, \mathcal{L}[q])$ which makes $T'_{\mathcal{S}}$ the exact inverse, we get $\Phi(\Phi(\mathcal{E})) \simeq T'_{\mathcal{S}}(T_{\mathcal{S}}(\mathcal{E})) \simeq \mathcal{E}$.

Conversely, assume $\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot H$, $H^2 = 2g - 2 \geq 2$, and $\mathcal{S}$ is torsion-free with Mukai vector $s = (s_0, s_1 H, s_2)$. The condition $\mathcal{S} \simeq R\mathcal{H}om(\mathcal{S}, \mathcal{L}[q])$ forces $q = 0$, so $\mathcal{S}$ is locally free, and $\mathcal{S} \simeq \mathcal{S}^\vee \otimes \mathcal{L}$. Taking the first Chern class yields $s_0 d = 2s_1$. The sphericity condition $s^2 = -2$ implies $2s_0 s_2 - s_1^2(2g - 2) = -2$. Substituting $s_1 = \frac{s_0 d}{2}$, we obtain:

$$s_0^2 d^2 (g - 1) = 4(s_0 s_2 - 1)$$

Thus $s_0 = 2^k$ for some integer $k \geq 0$ by mod p argument and $k = 0, 1$ by mod 16, so s_0 can only be 1 or 2. Finally, $s_0 d = 2s_1$ implies $\mathcal{L} \simeq \det(\mathcal{S})^{\frac{2}{s_0}}$. ☐

Lemma 2.3. *Let $p = 1$, assume that $\mathcal{S}$ and $\mathcal{E}$ are torsion-free sheaves, and the following conditions are satisfied:*

- $\mathrm{Ext}^1(\mathcal{S}, \mathcal{E}) = \mathrm{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$;
- *The cokernel of the evaluation map $e_{\mathcal{S},\mathcal{E}} : \mathrm{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \to \mathcal{E}$ is a torsion sheaf.*

Then $\Phi_{\mathcal{S},\mathcal{L}}(\mathcal{E})$ is a coherent sheaf and fits into the exact sequence:

$$0 \to \mathcal{E}^\vee \otimes \mathcal{L} \to \mathrm{Hom}(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{L} \to \Phi_{\mathcal{S},\mathcal{L}}(\mathcal{E}) \to \mathcal{E}xt^1(\mathcal{E}, \mathcal{L}) \to 0$$

Proof. Applying $R\mathcal{H}om(-, \mathcal{L})$ to the evaluation triangle $R\,\mathrm{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \to \mathcal{E} \to T_{\mathcal{S}}(\mathcal{E})$, and shifting by $+1$, yields the distinguished triangle:

$$R\mathcal{H}om(\mathcal{E}, \mathcal{L}) \to R\,\mathrm{Hom}(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{L} \to \Phi(\mathcal{E}) \xrightarrow{+1}$$

so for any integer k, we have:

$$\cdots \to \mathcal{E}xt^k(\mathcal{E}, \mathcal{L}) \to \mathrm{Ext}^k(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{L} \to \mathcal{H}^k(\Phi(\mathcal{E})) \to \mathcal{E}xt^{k+1}(\mathcal{E}, \mathcal{L}) \to \ldots$$

Since $\mathrm{coker}(e_{\mathcal{S},\mathcal{E}})$ is a torsion sheaf, $\mathcal{H}om(\mathrm{coker}(e_{\mathcal{S},\mathcal{E}}), \mathcal{L}) = 0$. Consequently, the transpose of the evaluation map, $\mathcal{H}om(\mathcal{E}, \mathcal{L}) \to \mathrm{Hom}(\mathcal{S}, \mathcal{E})^\vee \otimes \mathcal{S}^\vee \otimes \mathcal{L}$, is injective. Its kernel is $\mathcal{H}^{-1}(\Phi(\mathcal{E}))$, which forces $\mathcal{H}^{-1}(\Phi(\mathcal{E})) = 0$.

Furthermore, $\mathcal{E}$ is torsion-free on a surface, implying $\mathcal{E}xt^p(\mathcal{E},\mathcal{L}) = 0$ for $p \geq 2$. Combined with the hypothesis $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = \mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$, we deduce that $\mathcal{H}^p(\Phi(\mathcal{E})) = 0$ for all $p \geq 1$. Since its cohomology vanishes in all non-zero degrees, $\Phi(\mathcal{E}) \simeq \mathcal{H}^0(\Phi(\mathcal{E}))$ is a coherent sheaf.

Extracting the $k = 0$ segment of the long exact sequence directly provides the desired short exact sequence. □

Lemma 2.4. *Under assumption of 2.3, let $\mathcal{N} \in \mathrm{Pic}(X)$. Define a map $f : M(v) \to M(v')$ by $\mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{N}$ with $v' = v \cdot \mathrm{ch}(N)$. Set*

$$\mathcal{L}' = \mathcal{L} \otimes \mathcal{N}^{\otimes 2} \quad \mathcal{S}' = \mathcal{S} \otimes \mathcal{N}$$

Then the following diagram commutes:

$$\Phi_{\mathcal{S}',\mathcal{L}'} \circ f = f \circ \Phi_{\mathcal{S},\mathcal{L}}$$

And in particular, if $\Phi^p_{\mathcal{S},\mathcal{L}}(\mathcal{E}) \in M(v)$, then $\Phi^p_{\mathcal{S}',\mathcal{L}'}(\mathcal{E}') \in M(v')$

Proof. For any object $\mathcal{F} \in D^b(X)$, tensoring by a line bundle commutes with the spherical twist, meaning $T_{\mathcal{S}'}(\mathcal{F} \otimes \mathcal{N}) \simeq T_{\mathcal{S}}(\mathcal{F}) \otimes \mathcal{N}$. So

$$\Phi^p_{\mathcal{S}',\mathcal{L}'}(\mathcal{E} \otimes \mathcal{N}) = R\mathcal{H}om(T_{\mathcal{S}'}(\mathcal{E} \otimes \mathcal{N}), \mathcal{L}'[p]) \simeq R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{E}) \otimes \mathcal{N}, \mathcal{L} \otimes \mathcal{N}^{\otimes 2}[p])$$

Using the standard adjunction formula for derived tensor products, we obtain:

$$\begin{aligned}\Phi^p_{\mathcal{S}',\mathcal{L}'}(\mathcal{E} \otimes \mathcal{N}) &\simeq R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{E}), \mathcal{L} \otimes \mathcal{N}^{\otimes 2} \otimes \mathcal{N}^{\vee}[p]) \\ &\simeq R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{E}), \mathcal{L}[p]) \otimes \mathcal{N} \\ &= \Phi^p_{\mathcal{S},\mathcal{L}}(\mathcal{E}) \otimes \mathcal{N}\end{aligned}$$

For $p = 1$, if $\mathcal{E}$ satisfies the conditions such that $\Phi^1_{\mathcal{S},\mathcal{L}}(\mathcal{E})$ is a coherent sheaf in $M(v)$, then tensoring with $\mathcal{N}$ yields a coherent sheaf in $M(v')$. This establishes $\Phi_{\mathcal{S}',\mathcal{L}'}(\mathcal{E}') = \Phi_{\mathcal{S},\mathcal{L}}(\mathcal{E}) \otimes \mathcal{N} = f(\Phi_{\mathcal{S},\mathcal{L}}(f^{-1}(\mathcal{E}')))$. □

Remark 2.5. This lemma provides the numerical tool to check which moduli spaces $M(v)$ are preserved. To induce an automorphism on $M(v)$, we must solve the equation $v(\Phi(E)) = v$ for specific parameters d and $\mathcal{S}$.

Remark 2.6. As demonstrated by Lemma 2.4, when dealing with the line bundle $\mathcal{L} = \mathcal{O}_X(dH)$, we can assume $d \in \{0, 1\}$, so we restrict our attention to $d \in \{0, 1\}$ for the rest of the construction.

Furthermore, to make Φ to be an involution, by Lemma 2.2 (which restrict the rank of the torsion-free sheaf $\mathcal{S}$ to $s_0 \in \{1, 2\}$), we deduce that there are strictly two possibilities for the spherical object $\mathcal{S}$ and its Mukai vector s:

- $\mathcal{S} \simeq \mathcal{O}_X$, with Mukai vector $s = (1, 0, 1)$ and $d = 0$;
- $\mathcal{S}$ is a spherical stable bundle of rank 2, the genus g is even, its Mukai vector is $s = (2, H, \frac{g}{2})$, and $d = 1$.

These two disjoint cases represent the classical Markman-O'Grady/Beauville involution and the new involutions introduced in this paper, respectively.

2.2 Preserving the Mukai vector and semistability

There are 2 things we need to check to obtain an involution on $M(v)$, we need Φ is an automorphism of $M(v)$, means for $\mathcal{E} \in M(v)$, then $\Phi^1_{\mathcal{S},d}(\mathcal{E}) \in M(v)$.

- $v(\Phi(\mathcal{E})) = v$
- $\Phi(\mathcal{E})$ is torsion-free and semi-stable.

Here we assume $\mathcal{E}, \mathcal{S}$ are torsion free sheaves and $v(\mathcal{E}) = (v_0, v_1 H, v_2)$ and $v(\mathcal{S}) = (s_0, s_1 H, s_2)$. The following result is used to compute $v(\Phi(\mathcal{E}))$.

Lemma 2.7. *The action of Φ on Mukai vectors satisfies:*

$$v(\Phi(\mathcal{E})) = -(D \circ R_{\mathcal{S}}(v(\mathcal{E}))) \cdot \mathrm{ch}(\mathcal{O}_X(dH))$$

where D is the duality morphism $D(w_0, c_1, w_2) = (w_0, -c_1, w_2)$ and $R_{\mathcal{S}}$ is the reflection along the hyperplane orthogonal to $v(\mathcal{S})$.

The following result is $v = (v_0, v_1 H, v_2)$ fixed under action of Φ.

Lemma 2.8. *Let v be a Mukai vector with $s \cdot v \neq 0$. Then $v(\Phi(\mathcal{E})) = v$ if and only if:*

- $(s_0, d) = (1, 0)$*, $\mathcal{S} = \mathcal{O}_X$ and $v_0 = v_2$; or*
- $(s_0, d) = (2, 1)$*, $\mathcal{S}$ is a spherical bundle with $s = (2, H, g/2)$ and*

$$2v_2 = (2g-2)v_1 - v_0\left(\frac{g}{2} - 1\right)$$

Proof. By 2.7, the action of Φ on the Mukai vector is given by $v(\Phi(\mathcal{E})) = -(D \circ R_{\mathcal{S}}(v(\mathcal{E}))) \cdot \mathrm{ch}(\mathcal{O}_X(dH)) = (w_0, w_1 H, w_2)$.

Let us compute this explicitly. Let $v = (v_0, v_1 H, v_2)$ and $s = (s_0, s_1 H, s_2)$. We denote $a := -s \cdot v$, by explicit computation, we obtain:

$$\begin{aligned} w_0 &= v_0 - a s_0 \\ w_1 &= -(v_1 - a s_1) + d(v_0 - a s_0) \\ w_2 &= v_2 - a s_2 - d(v_1 - a s_1)(2g-2) + d^2(g-1)(v_0 - a s_0) \end{aligned}$$

so $v(\Phi(\mathcal{E})) = v$ yields the following system of equations:

$$\begin{aligned} 2v_0 &= a s_0 \\ a s_1 &= -d(v_0 - a s_0) = d v_0 \\ v_2 &= -v_2 + a s_2 + d(v_1 - a s_1)(2g-2) - d^2(g-1)(v_0 - a s_0) \end{aligned}$$

From Lemma 2.4, we know that the only valid spherical bundles and twists leading to an involution on a K3 surface of Picard rank 1 correspond to either $s_0 = 1, d = 0$ or $s_0 = 2, d = 1$, direct computation to each case shows

- *Case I:* $(s_0 = 1, d = 0)$ The first case gives $\mathcal{S} = \mathcal{O}_X$ and $v_0 = v_2$
- *Case II:* $(s_0 = 2, d = 1)$ Direct computation shows $v_0 = a$ and

$$\begin{aligned} 2v_2 &= a s_2 + d(v_1 - a s_1)(2g-2) + d^2(g-1)v_0 \\ &= (2g-2)v_1 - v_0\left(\frac{g}{2} - 1\right) \end{aligned}$$

□

Now we prove that Φ also preserves semistability of torsion free sheaves, here we change the assumption a little to set $\mathrm{Pic}(X) = \mathbb{Z} \cdot H \oplus^{\perp} N$ with H ample divisor and $H^2 = 2g - 2$, N is a lattice that does not contain any effective divisor.

Proposition 2.9. *Assume that* $\mathrm{Ext}^1(\mathcal{S}, \mathcal{E}) = \mathrm{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$*. Let* $e_{\mathcal{S},\mathcal{E}}$ *be the evaluation map.*

- *If* $\dim(\mathrm{coker}(e_{\mathcal{S},\mathcal{E}})) = 0$*, then* $\Phi(\mathcal{E})$ *is a torsion-free sheaf.*
- *If in addition* $\mathrm{Ker}(e_{\mathcal{S},\mathcal{E}})$ *is slope-stable, then* $\Phi(\mathcal{E})$ *is slope-stable.*

Proof. Set $\mathcal{K} := \mathrm{Ker}(e_{\mathcal{S},\mathcal{E}})$ and $\mathcal{E}_0 := \mathrm{Im}(e_{\mathcal{S},\mathcal{E}})$. We have the exact sequence:

$$0 \to \mathcal{K} \to \mathrm{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \to \mathcal{E}_0 \to 0 \tag{1}$$

Since $\mathrm{coker}(e_{\mathcal{S},\mathcal{E}})$ supported on isolated points, we have $c_1(\mathcal{E}_0) = c_1(\mathcal{E})$ and $\mathcal{E}^\vee \simeq \mathcal{E}_0^\vee$. As $\mathcal{K}$ is the kernel of a map between torsion-free sheaves on a surface, it is a reflexive sheaf, and hence locally free. We can compute its first Chern class:

$$c_1(\mathcal{K}) = -(v(\mathcal{S}) \cdot v)c_1(\mathcal{S}) - c_1(\mathcal{E})$$

By Lemma 2.3, $\Phi(\mathcal{E})$ is a coherent sheaf. Let $\mathcal{I}$ be the image of the middle map in the exact sequence for $\Phi(\mathcal{E})$. We obtain the short exact sequence:

$$0 \to \mathcal{I} \to \Phi(\mathcal{E}) \to \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X(dH)) \to 0 \tag{2}$$

Because $\mathcal{E}$ is torsion-free, the term $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X(dH))$ is supported on isolated points.

Now, if we dualize the sequence (1) and tensor it by $\mathcal{O}_X(dH)$, we get:

$$0 \to \mathcal{E}_0^\vee(dH) \to \mathrm{Hom}^\vee \otimes \mathcal{S}^\vee(dH) \to \mathcal{K}^\vee(dH) \to \mathcal{E}xt^1(\mathcal{E}_0, \mathcal{O}_X(dH)) \to 0 \tag{3}$$

Notice that because $\mathcal{E}_0^\vee \simeq \mathcal{E}^\vee$, the first map in (3) is isomorphic to the first map defining $\mathcal{I}$, Therefore, $\mathcal{I} \simeq \mathcal{K}^\vee(dH)$. From (2), $\Phi(\mathcal{E})$ and $\mathcal{I}$ differ only along a zero-dimensional subset, it gives $\Phi(\mathcal{E})^\vee \simeq \mathcal{I}^\vee \simeq \mathcal{K}(-dH)$. Comparing first Chern classes, this gives $c_1(\Phi(\mathcal{E})^\vee) = c_1(\mathcal{K}) - \mathrm{rk}(\mathcal{K})dH = -c_1(\Phi(\mathcal{E}))$.

If $\Phi(\mathcal{E})$ had a 1-dimensional torsion subsheaf T, the dual of $\Phi(\mathcal{E})$ would be exactly the dual of its torsion-free quotient, yielding $c_1(\Phi(\mathcal{E})^\vee) = -c_1(\Phi(\mathcal{E})) + c_1(T)$. Since we found $c_1(\Phi(\mathcal{E})^\vee) = -c_1(\Phi(\mathcal{E}))$, it must be that $c_1(T) = 0$, meaning $\Phi(\mathcal{E})$ has no 1-dimensional torsion.

Furthermore, because Φ is an anti-auto-equivalence, $\mathrm{Ext}^i(\Phi(\mathcal{E}), \Phi(\mathcal{E})) \simeq \mathrm{Ext}^i(\mathcal{E}, \mathcal{E})$. The stability of $\mathcal{E}$ implies it is simple, so $\Phi(\mathcal{E})$ is also simple. If $\Phi(\mathcal{E})$ had a 0-dimensional torsion subsheaf, it would admit an epimorphism to a skyscraper sheaf $\mathcal{O}_p$. Composing this with the natural inclusion of $\mathcal{O}_p$ back into the torsion part of $\Phi(\mathcal{E})$ would create a non-scalar nilpotent endomorphism of $\Phi(\mathcal{E})$, contradicting its simplicity. Thus, (i) is proved.

For part (ii), if $\mathcal{K}$ is a slope-stable vector bundle, then $\mathcal{K}^\vee(dH)$ is also slope-stable. Since $\mathcal{I} \simeq \mathcal{K}^\vee(dH)$, $\mathcal{I}$ is slope-stable. According to sequence (2), $\Phi(\mathcal{E})$ and $\mathcal{I}$ differ only on a zero-dimensional scheme. Hence, $\Phi(\mathcal{E})$ is slope-stable. □

Proposition 2.10. *Let* $\mathcal{S}$ *be a spherical bundle with Mukai vector* $(2, H, \frac{g}{2})$ *and* g *even. Assume the following conditions hold:*

- $\mathrm{Ext}^1(\mathcal{S}, \mathcal{E}) = \mathrm{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$ *and* $\mathrm{Hom}(\mathcal{S}, \mathcal{E}) \neq 0$*;*

- $\mathrm{rk}(\mathcal{E}) = 2k+1$ *with* $k \in \mathbb{N}$*;*
- $v(\Phi(\mathcal{E})) = v(\mathcal{E})$ *and* $c_1(\mathcal{E}) = (k+1)H$*;*
- $\mathrm{Pic}(X) = \mathbb{Z} \cdot H \oplus^{\perp} N$ *with* N *containing no effective divisors.*

Then the cokernel of the evaluation map $\mathrm{coker}(e_{\mathcal{S},\mathcal{E}})$ *is supported on isolated points, and* $\ker(e_{\mathcal{S},\mathcal{E}})$ *is slope-stable.*

Proof. Let $\mathcal{E}_0 = \mathrm{Im}(e_{\mathcal{S},\mathcal{E}})$. Suppose $c_1(\mathcal{E}_0) = cH + L$ for some $L \in N$ and $c \in \mathbb{Z}$. By the stability of $\mathcal{S}$ and $\mathcal{E}$, the slope satisfies $\mu(\mathcal{S}) \leq \mu(\mathcal{E}_0) \leq \mu(\mathcal{E})$, meaning:

$$\frac{1}{2} \leq \frac{c}{\mathrm{rk}(\mathcal{E}_0)} \leq \frac{k+1}{2k+1}$$

Since $\mathrm{rk}(\mathcal{E}_0) \leq \mathrm{rk}(\mathcal{E}) = 2k+1$, the strict inequality $\frac{1}{2} < \frac{c}{\mathrm{rk}(\mathcal{E}_0)} < \frac{k+1}{2k+1}$ implies

$$\frac{\mathrm{rk}(\mathcal{E}_0)}{2} < c < \frac{\mathrm{rk}(\mathcal{E}_0)}{2} + \frac{\mathrm{rk}(\mathcal{E}_0)}{4k+2} \leq \frac{\mathrm{rk}(\mathcal{E}_0)}{2} + \frac{1}{2}$$

which is impossible for integers c and $\mathrm{rk}(\mathcal{E}_0)$.

Now we have the following 2 cases:

- *Case i,* $\frac{c}{\mathrm{rk}(\mathcal{E}_0)} = \frac{1}{2}$*:* This means $\mathrm{rk}(\mathcal{E}_0) = 2c$. Let $\mathcal{K} = \mathrm{Ker}(e_{\mathcal{S},\mathcal{E}})$. First we know

  $$\dim \mathrm{Hom}(\mathcal{S}, \mathcal{E}) = \chi(\mathcal{S}^{\vee} \otimes \mathcal{E}) = -s \cdot v = 2k+1$$

  by Riemann-Roch theorem and the vanishing conditions $\mathrm{Ext}^1 = \mathrm{Ext}^2 = 0$.

  From the exact sequence $0 \to \mathcal{K} \to \mathcal{S}^{\oplus(2k+1)} \to \mathcal{E}_0 \to 0$, we know $\mathrm{rk}(\mathcal{K}) = 4k+2-2c$ and $c_1(\mathcal{K}) = (2k+1-c)H$ So the slope is $\mu(\mathcal{K}) = \frac{1}{2} = \mu(\mathcal{S}^{\oplus(2k+1)})$, meaning $\mathcal{K} \simeq \mathcal{S}^{\oplus\alpha}$.

  Now, apply the functor $\mathrm{Hom}(\mathcal{S}, -)$ to the exact sequence, we know

  $$0 \to \mathrm{Hom}(\mathcal{S}, \mathcal{K}) \to \mathrm{Hom}(\mathcal{S}, \mathcal{S}^{\oplus(2k+1)}) \to \mathrm{Hom}(\mathcal{S}, \mathcal{E}_0)$$

  Since $\mathcal{E}_0$ is the natural image of the evaluation map, the induced rightmost map is an isomorphism, implies $\mathrm{Hom}(\mathcal{S}, \mathcal{K}) = 0$, then $\alpha = 0$, forcing $\mathrm{rk}(\mathcal{E}_0) = 4k+2$, which absurdly exceeds $\mathrm{rk}(\mathcal{E}) = 2k+1$.

- *Case ii,* $\frac{c}{\mathrm{rk}(\mathcal{E}_0)} = \frac{k+1}{2k+1}$*:* Notice that the integers $k+1$ and $2k+1$ are coprime, the denominator $\mathrm{rk}(\mathcal{E}_0)$ must be an integer multiple of $2k+1$, and $\mathcal{E}_0 \subset \mathcal{E}$ means

  $$\mathrm{rk}(\mathcal{E}_0) = \mathrm{rk}(\mathcal{E}) \quad \text{and} \quad c = k+1$$

  Consequently, $\mathcal{E}_0$ and $\mathcal{E}$ share the exact same rank and the same first Chern class, so $\mathrm{coker}(e_{\mathcal{S},\mathcal{E}}) \simeq \mathcal{E}/\mathcal{E}_0$ has its rank and first Chern class 0, means it is supported on isolated points.

To prove that $\mathcal{K} = \mathrm{Ker}(e_{\mathcal{S},\mathcal{E}})$ is slope-stable, we proceed by contradiction. Assume there exists a proper destabilizing subsheaf $\mathcal{H} \subset \mathcal{K}$. From the previous steps, we can compute the slope of $\mathcal{K}$ is exactly $\mu(\mathcal{K}) = \frac{k}{2k+1}$.

Since $\mathcal{H}$ is assumed to destabilize $\mathcal{K}$, we must have $\mu(\mathcal{H}) \geq \mu(\mathcal{K})$. Let $c_1(\mathcal{H}) = \alpha H + D'$ (where $D' \in N$). Then $\mu(\mathcal{H}) = \frac{\alpha}{\mathrm{rk}(\mathcal{H})}$.

Furthermore, because $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{K} \subset \mathrm{Hom}(\mathcal{S}, \mathcal{E}) \otimes \mathcal{S} \simeq \mathcal{S}^{\oplus(2k+1)}$, we know

$$\frac{1}{2} - \frac{1}{4k+2} = \frac{k}{2k+1} \leq \frac{\alpha}{\mathrm{rk}(\mathcal{H})} \leq \frac{1}{2}$$

By $\mathrm{rk}(\mathcal{H}) < 2k+1$ since $\mathcal{H}$ is a proper subsheaf of $\mathcal{K}$, there is a strict arithmetic bounds:

$$\frac{\mathrm{rk}(\mathcal{H})}{2} - \frac{1}{2} < \alpha \leq \frac{\mathrm{rk}(\mathcal{H})}{2}$$

Since $\alpha \in \mathbb{N}$, the only possible solution is $\alpha = \frac{\mathrm{rk}(\mathcal{H})}{2}$, which forces $\mathrm{rk}(\mathcal{H}) = 2\alpha$. This strictly forces $\mu(\mathcal{H}) = 1/2$ and $\mathcal{H} \simeq \mathcal{S}^{\oplus\alpha}$.

However, the inclusion $\mathcal{H} \hookrightarrow \mathcal{K}$ would then provide a non-zero morphism from $\mathcal{S}$ to $\mathcal{K}$. This outright contradicts our earlier finding that $\mathrm{Hom}(\mathcal{S}, \mathcal{K}) = 0$. That completes the proof. □

3 Involutions on K3 surfaces

3.1 Classical examples of involutions

The following constructions describe birational involutions on moduli spaces of sheaves (or Hilbert schemes) without relying on derived category auto-equivalences, but rather through classical projective geometry.

3.1.1 The Beauville Involution ($g = 3, r = 1, n = 2$)

Let $X \subset \mathbb{P}^3$ be a smooth quartic K3 surface, hence $g = 3$. The moduli space $M(1, H, 1)$ is isomorphic to the Hilbert scheme of two points, $X^{[2]}$, with isomorphism given by $\mathcal{E} \mapsto Z = \mathrm{supp}(\mathcal{O}_X(H)/\mathcal{E})$.

- Choose a general zero-dimensional subscheme $Z \in X^{[2]}$, which geometrically represents two distinct points on X.
- These two points span a unique line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^3$.
- By Bézout's Theorem, the intersection of the line L with the quartic surface X is a zero-dimensional scheme of length 4 (i.e., 4 points).
- Therefore, $L \cap X = Z \cup Z'$, where Z' is the residual subscheme of length 2.
- The birational involution is given by the residual map $Z \mapsto Z'$.

3.1.2 The Markman-O'Grady Involution ($g = 5, r = 2$)

Let $X \subset \mathbb{P}^5$ be a smooth K3 surface of genus $g = 5$, classically constructed as the complete intersection of three quadratic hypersurfaces. Consider the moduli space $M_X(2, H, 2)$ of rank 2 stable sheaves.

- The linear system of quadrics in $\mathbb{P}^5$ containing X is parameterized by a projective plane $\mathbb{P}^2$.
- A general point in this $\mathbb{P}^2$ corresponds to a smooth 4-dimensional quadrtic hypersurface Q. It is a well-known fact that any such quadric Q carries exactly two distinct spinor bundles of rank 2, say S_1 and S_2.
- Restricting these spinor bundles to X provides two stable rank 2 vector bundles, $E_1 = S_1|_X$ and $E_2 = S_2|_X$, which represent two distinct points in $M_X(2, H, 2)$.

- The moduli space $M_X(2, H, 2)$ is birational to a double cover of the plane $\mathbb{P}^2$ branched over the sextic curve of singular quadrics. The Markman-O'Grady involution acts by exchanging the two sheets of this cover, mapping $E_1 \mapsto E_2$ and $E_2 \to E_1$, which correspond to swapping the two spinor bundles associated with the same quadric.

3.1.3 The Beri-Manivel Involution ($g = 10, n = 3$)

Let X be a general K3 surface of genus $g = 10$. By Mukai's classification, X can be embedded into the Grassmannian $\mathrm{Gr}(2, 7)$ by considering the natural spherical bundle $\mathcal{S}$ with $c_1(\mathcal{S}) = H$, and $H^0(X, \mathcal{S}) = 7$ by Riemann-Roch theorem, this is related to $\mathrm{Im}(\mathbb{O})$ where $\mathbb{O}$ is octonions and representation theory of Lie group G_2. We consider the Hilbert scheme of three points, $X^{[3]}$.

- Choose a general subscheme $Z = \{x_1, x_2, x_3\} \in X^{[3]}$.
- Under the embedding $X \hookrightarrow \mathrm{Gr}(2, 7)$, each point $x_i \in X$ corresponds to a 2-dimensional vector subspace (a plane) $P_i \subset \mathbb{C}^7$.
- For a generic choice of three points, the corresponding three planes P_1, P_2, P_3 are independent and therefore span a 6-dimensional subspace $V_6 \subset \mathbb{C}^7$.
- This V_6 naturally defines a smaller Grassmannian $\mathrm{Gr}(2, V_6) \subset \mathrm{Gr}(2, 7)$. The intersection of this sub-Grassmannian with the K3 surface, $X \cap \mathrm{Gr}(2, V_6)$, is a zero-dimensional scheme of length $c_2(\mathrm{Gr}(2, 6)) = 6$.
- Since this intersection already contains the initial length 3 scheme Z, we can write $X \cap \mathrm{Gr}(2, V_6) = Z \cup Z'$, where Z' is the residual subscheme of length 3.
- The birational involution defined by Beri and Manivel is exactly the mapping $Z \mapsto Z'$.

3.2 Reconstruct by derived category method

To establish the regularity of the map and prove it is more than just a birational transformation, we first need to ensure the vanishing of the relevant cohomology group for every point in the moduli space.

Lemma 3.1. *Assume $r^2 \leq g < (r+1)^2$. Let $\mathcal{E}$ be any element of $M(r, H, r)$. Then $H^1(\mathcal{E}) = 0$.*

Remark 3.2. The regularity of the Markman-O'Grady involution relies heavily on proving $H^1(\mathcal{E}) = 0$ (which implies $\mathrm{Ext}^1(\mathcal{E}, \mathcal{O}_X) = 0$ by Serre duality) for any $\mathcal{E} \in M(r, H, r)$. The proof proceeds by a contradiction:

- Assume $H^1(\mathcal{E}) \neq 0$, which means $\mathrm{Ext}^1(\mathcal{E}, \mathcal{O}_X) \neq 0$. This non-vanishing allows us to construct a non-trivial extension sequence: $0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{E} \to 0$.
- We compute the Mukai vector of $\mathcal{F}$ as $v(\mathcal{F}) = (r + 1, H, r + 1)$. Its self-intersection is:

  $$v(\mathcal{F})^2 = 2(g - (r + 1)^2) - 2$$

  This is exactly where the upper bound $g < (r + 1)^2$ is used, it forces $v(\mathcal{F})^2 \leq -4 < -2$.
- A fundamental theorem on K3 surfaces states that any stable sheaf must satisfy $v^2 \geq -2$, so $\mathcal{F}$ must contain a saturated destabilizing subsheaf $\mathcal{K}$.
- A pure homological algebra argument, the rigid stability of $\mathcal{E}$ and $\mathcal{O}_X$ forces $\mathcal{K}$ to essentially equal $\mathcal{E}$. This induces an inverse mapping $\mathcal{E} \to \mathcal{F}$, meaning the extension sequence splits. So we obtain $H^1(\mathcal{E}) = 0$.

With this vanishing result in hand, which ensures that the image of the functor is always a coherent sheaf and not a complex, we can formally state the existence of the biregular involution.

Proposition 3.3. *Let X be a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H \oplus^{\perp} N$, $H^2 = 2g-2$, and N not containing any effective divisor. Let $r \geq 1$ be an integer such that $r^2 \leq g < (r+1)^2$. Then, whenever $\mathcal{E}$ lies in $M(r, H, r)$, its image $\Phi(\mathcal{E})$ also lies in $M(r, H, r)$. Consequently, the auto-equivalence Φ induces a biregular involution on $M(r, H, r)$.*

Proof. By Lemma 2.8, we already know that the action on the Mukai vector satisfies $v(\Phi(\mathcal{E})) = v(\mathcal{E})$. So, it remains to prove that $\Phi(\mathcal{E})$ is slope-stable and torsion-free.

This is a direct consequence of Proposition 2.9 and Lemma 2.10 if we know that $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{E}) = \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{E}) = 0$. However, the first vanishing is exactly the content of Lemma 3.1, while the second vanishing follows naturally from the stability of $\mathcal{E}$ (since $\mathrm{Ext}^2(\mathcal{O}_X, \mathcal{E}) \simeq \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)^{\vee} = 0$ for stable sheaves with appropriate slopes). □

Now we are ready to reconstruct the Markman-O'Grady's involution.

Example 3.4. Let (X, H) be a K3 surface of genus $g = 5$, embedded as the intersection of three quadratic hypersurfaces in $\mathbb{P}^5$. The moduli space $M(2, H, 2)$ is a K3 surface of genus 2, which admits a double cover $\pi : M(v) \to \mathbb{P}^2$ branched over a sextic curve.

- The plane $\mathbb{P}^2$ parametrizes the web of quadrics in $\mathbb{P}^5$ containing X.
- For a non-singular quadric Q in the web, there are two distinct spinor bundles $\mathcal{G}$ and $\mathcal{G}'$ of rank 2. These bundles fit into the following short exact sequence on Q:

$$0 \to \mathcal{G}^{\vee} \to \mathcal{O}_Q^{\oplus 4} \to \mathcal{G}' \to 0$$

- Restricting this sequence to the K3 surface $X \subset Q$, we observe that the restriction $\mathcal{G}|_X$ and $\mathcal{G}'|_X$ correspond to conjugate points in $M(v)$ under the double cover π.
- Comparing this with the categorical definition of the functor $\Phi = \Phi_{\mathcal{O}_X,0}$, which is based on the evaluation triangle $R\,\mathrm{Hom}(\mathcal{O}_X, \mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E} \to T_{\mathcal{O}_X}(\mathcal{E})$, the resulting sheaf satisfies:

$$\mathcal{G}'|_X \simeq \Phi(\mathcal{G}|_X)$$

Thus, the categorical involution Φ acts precisely by exchanging the two sheets of the double cover, effectively swapping the restricted spinor bundles associated with the same quadric Q.

Remark 3.5. Here we list the full detail of $\mathcal{G}'|_X \simeq \Phi(\mathcal{G}|_X)$, where $\Phi = \Phi_{\mathcal{O}_X,0}$, the first exact sequence comes from restircting

$$0 \to \mathcal{G}^{\vee}|_X \to \mathcal{O}_X^{\oplus 4} \to \mathcal{G}'|_X \to 0$$

and using

$$0 \to \mathcal{E}^{\vee} \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{E})^{\vee} \otimes \mathcal{O}_X \to \Phi(\mathcal{E}) \to \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X) \to 0$$

- $\mathcal{E}^{\vee} = (\mathcal{G}|_X)^{\vee} = \mathcal{G}^{\vee}|_X$ by definition.
- The Mukai vector of $\mathcal{E}$ is $v = (2, H, 2)$, so $\chi(\mathcal{E}) = 2 + 2 = 4$. Since $\mathcal{E}$ is stable of positive slope, $H^2(\mathcal{E}) = \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)^{\vee} = 0$, combined with $H^1(\mathcal{E}) = 0$, we get exactly $\mathrm{Hom}(\mathcal{O}_X, \mathcal{E})^{\vee} \otimes \mathcal{O}_X \simeq \mathcal{O}_X^{\oplus 4}$.
- Since $\mathcal{E}$ is a restriction of a vector bundle, it is locally free on the smooth K3 surface X. The local Ext-sheaf $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_X)$ strictly vanishes for locally free sheaves. Thus, the rightmost term is 0.

then comparing these 2 sequence obtains the isomorphism between $\mathcal{G}'|_X$ and $\Phi(\mathcal{G}|_X)$.

The geometric constructions described above are deeply classical, relying heavily on projective geometry and intersection theory. However, the true power of the categorical approach in [FMM25] is its ability to blindly reconstruct these exact geometric mappings using purely homological algebra. Let us see how the derived functors intrinsically process these geometric residuals.

Example 3.6. Let $X \subset \mathbb{P}^3$ be a quartic K3 surface. We aim to rigorously prove that the categorical functor $\Phi = \Phi_{\mathcal{O}_X,0}$ correctly reproduces the map $Z \mapsto Z'$ on $M(1,H,1) \cong X^{[2]}$. Let $\mathcal{E} := \mathcal{I}_Z(H)$ be our initial stable sheaf.

The Mukai vector is $v = (1,H,1)$. By Riemann-Roch, $\chi(\mathcal{E}) = \chi(\mathcal{O}_X(H)) - \text{length}(Z) = 4-2 = 2$. By Lemma 3.1, $H^1(\mathcal{E}) = 0$, and stability implies $H^2(\mathcal{E}) = 0$. Thus, $h^0(\mathcal{E}) = 2$, giving $\text{Hom}(\mathcal{O}_X,\mathcal{E}) \cong \mathbb{C}^2$.

Following Lemma 2.3, since $\text{Ext}^1(\mathcal{O}_X,\mathcal{E}) = 0$, the image $\Phi(\mathcal{E})$ is a pure coherent sheaf fitting into the fundamental sequence:

$$0 \to \mathcal{E}^\vee \to \text{Hom}(\mathcal{O}_X,\mathcal{E})^\vee \otimes \mathcal{O}_X \to \Phi(\mathcal{E}) \to \mathcal{E}xt^1(\mathcal{E},\mathcal{O}_X) \to 0 \tag{4}$$

We now explicitly compute the outer terms of (4):

- Since Z is a codimension-2 subscheme on a smooth surface, its ideal sheaf satisfies $\mathcal{I}_Z^\vee \cong \mathcal{O}_X$. Therefore, $\mathcal{E}^\vee = (\mathcal{I}_Z \otimes \mathcal{O}_X(H))^\vee \cong \mathcal{O}_X(-H)$.
- Applying the functor $\mathcal{H}om(-,\mathcal{O}_X)$ to $0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z \to 0$, local duality on the K3 surface dictates that $\mathcal{E}xt^1(\mathcal{I}_Z(H),\mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_Z,\mathcal{O}_X(-H)) \cong \mathcal{O}_Z$.

Substituting these into (4), the categorical sequence becomes:

$$0 \to \mathcal{O}_X(-H) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2} \to \Phi(\mathcal{E}) \to \mathcal{O}_Z \to 0 \tag{5}$$

Geometrically, the map $\alpha : \mathcal{O}_X(-H) \to \mathcal{O}_X^{\oplus 2}$ is defined by the two global sections of $\mathcal{I}_Z(H)$. These sections represent two hyperplanes in $\mathbb{P}^3$ passing through Z. Their intersection is exactly the secant line L. The base locus of this pencil on X is $W = L \cap X$, standard results of Koszul complex gives

$$0 \to \mathcal{O}_X(-H) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2} \to \mathcal{I}_W(H) \to 0$$

then we can get

$$0 \to \mathcal{I}_W(H) \to \Phi(\mathcal{E}) \to \mathcal{O}_Z \to 0$$

Now, let us look at the pure geometry, the line L intersects the quartic surface X at a length-4 scheme $W = Z \cup Z'$, where Z' is the residual scheme of length 2. So $W \supset Z'$ gives:

$$0 \to \mathcal{I}_W(H) \to \mathcal{I}_{Z'}(H) \to \mathcal{O}_Z \to 0$$

(This arises because $\mathcal{I}_{Z'}/\mathcal{I}_W \cong \mathcal{O}_W/\mathcal{O}_{Z'} \cong \mathcal{O}_Z$). Then comparing (4) with

$$0 \to \mathcal{O}_X(-H) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2} \to \mathcal{I}_{Z'}(H) \to \mathcal{O}_Z \to 0$$

harvest the result we want.

The exact same categorical philosophy applies to the much more complex Beri-Manivel involution, shifting from a rank-1 twist to a rank-2 twist. So before we start, we need to clarify the settings of rank-2 twist.

3.3 New constructions via rank-2 spherical bundles

Theorem 3.7. *Let $(g,k) \in \mathbb{N}^2$ with $g \equiv 2 \pmod 4$. Let (X,H) be a polarized K3 surface such that $\mathrm{Pic}(X) = \mathbb{Z} \cdot H \oplus^{\perp} N$ with $H^2 = 2g-2$ and N not containing any effective divisor. Let $\mathcal{S}$ be the spherical bundle with Mukai vector $v(\mathcal{S}) = (2, H, g/2)$. Consider the Mukai vector $v = (v_0, v_1, v_2)$ with:*

- $v_0 = 2k+1$,
- $v_1 = (k+1)H$,
- $2v_2 = (2g-2)(k+1) - (2k+1)(\frac{g}{2} - 1)$.

We assume that $\dim(M(v)) \geq 2$, namely $g \geq (2k+1)^2 + 1$. Then the functor $\Phi_{\mathcal{S},1}$ induces a birational involution on the moduli space $M(v)$.

Remark 3.8. The proof fundamentally relies on establishing the following properties for a generic sheaf $\mathcal{E} \in M(v)$:

- *Vanishing of Ext groups:* The most technically demanding step is to show $\mathrm{Ext}^1(\mathcal{S},\mathcal{E}) = 0$, this is achieved by analyzing generic non-splitting extensions via [FMM25, Lemmas 4.5, 4.6]. Furthermore, $\mathrm{Ext}^2(\mathcal{S},\mathcal{E}) = 0$ follows naturally from the stability of both $\mathcal{S}$ and $\mathcal{E}$.
- *Non-trivial Homomorphisms:* Using the Riemann-Roch theorem, one computes the Euler characteristic to ensure that $\dim \mathrm{Hom}(\mathcal{S},\mathcal{E}) = 2k+1 \neq 0$.
- *Preservation of Stability:* Once the cohomological conditions above are satisfied, one directly applies the foundational results from Lemma 2.8 and Propositions 2.9, 2.10 to guarantee that $v(\Phi(\mathcal{E})) = v$ and $\Phi(\mathcal{E})$ remains slope-stable and torsion-free.

Remark 3.9. Theorem 3.7 is the central result of this construction, as it provides a unified derived-category framework that naturally generates and generalizes classical geometric involutions:

- *The Beri-Manivel Involution:* By setting $k=0$, the theorem yields birational involutions on the Hilbert scheme of points $X^{[n]}$ with $n = \frac{g+2}{4}$. Specifically, for a K3 surface of genus $g = 10$ ($n=3$), this exactly recovers the geometric involution on $X^{[3]}$ recently discovered by Beri and Manivel using Grassmannian geometry.
- *Markman-O'Grady Involutions:* While the Markman-O'Grady involutions are obtained by twisting around the rank-1 structure sheaf ($\mathcal{S} = \mathcal{O}_X$ with $d=0$), Theorem 3.7 successfully extends this exact same functorial philosophy to rank-2 spherical bundles ($d=1$). It provides a parallel, higher-rank construction mechanism, showing that both Beri-Manivel and Markman-O'Grady involutions are simply two facets of the same underlying categorical phenomenon.

3.4 Anti-symplectic involution

Proving the anti-symplectic nature of the involution is the first crucial step in understanding its geometric essence. We begin by recalling O'Grady's result on the classical case, which serves as a benchmark for our subsequent discussion.

Let $M(v)$ be a compact moduli space of stable sheaves on X of dimension ≥ 4, we know there is an Hodge isometry

$$\theta : v^{\perp} \subset H^*(X,\mathbb{Z}) \to H^2(M(v),\mathbb{Z})$$

where $H^*(X,\mathbb{Z})$ is endowed with the Mukai pairing and $H^2(M(v),\mathbb{Z})$ is endowed with Bogomolov-Beauville-Fujiki form.

Proposition 3.10. *Let (X, H) be a polarized K3 surface such that* $\mathrm{Pic}(X) = \mathbb{Z} \cdot H \oplus^{\perp} N$*, with $H^2 = 2g - 2$ and N not containing effective divisors. Let $v = (r, H, r)$ be a Mukai vector with $r \geq 2$ and* $\dim M(v) \geq 4$*. Let $h_v := \theta(1, 0, -1)$ and R_{h_v} be the reflection associated with the Beauville-Bogomolov form. Then the involution $\Phi_{\mathcal{O}_X,0}$ on $M(v)$ is anti-symplectic, and its action on $H^2(M(v), \mathbb{Z})$ is given by $\Phi^*_{\mathcal{O}_X,0} = R_{h_v}$.*

Having established the result for the rank 1 case, we now need to compute the local action on cohomology for the newly constructed involution $\Phi_{\mathcal{S},1}$, which is based on a rank 2 spherical bundle.

Lemma 3.11. *We assume that $U(v)$ is a non-empty Zariski open set in $M(v)$. Let $\iota_v : U(v) \hookrightarrow M(v)$ be the embedding. The action of $\Phi_{\mathcal{S},1}$ on $H^2(M(v), \mathbb{Z})$ satisfies:*

$$\iota_v^* \circ \Phi^*_{\mathcal{S},1} = \iota_v^* \circ R_{d_v}$$

where $d_v := \theta(2, H, \frac{g}{2} - 1)$ and R_{d_v} is the reflection given by $R_{d_v}(\alpha) = -\alpha + B_{M(v)}(d_v, \alpha) d_v$.

Remark 3.12. The proof fundamentally relies on the Hodge isometry θ above and the Riemann-Roch theorem. By tracking the Chern character of the transformed universal family, the functor's action translates to a reflection R_{d_v} on the Beauville-Bogomolov lattice. Since d_v is purely algebraic, it is strictly orthogonal to the transcendental lattice $T_{M(v)}$, meaning the reflection acts as $-id$ on $T_{M(v)}$, which proves the anti-symplectic nature.

However, because our involution is birational, we must carefully consider its domain of definition, which leads us to the following geometric subtlety.

Remark 3.13. The set $M(v) \backslash U(v)$ where the involution is undefined could have codimension 1. Although the involution Φ can always be extended to a bimeromorphism regular in codimension 2 [MR20, Lemma 3.2], its induced action on the full cohomology $H^2(M(v), \mathbb{Z})$ might differ from the local action computed in Lemma 3.11.

In the case $M(v) \setminus U(v)$ is sufficiently small, the global cohomological action can be explicitly and completely determined.

Remark 3.14. If the complement $M(v) \setminus U(v)$ has codimension at least 2, the action of Φ^* on the entire $H^2(M(v), \mathbb{Z})$ is globally given by $\Phi^* = \theta \circ R_{\mathcal{S}} \circ D \circ T \circ \theta^{-1}$.

Furthermore, comparing this with the action derived from Mukai vectors yields an alternative expression $\varphi = -(\theta \circ T^{-1} \circ D \circ R_{\mathcal{S}} \circ \theta^{-1}) = -(\Phi^*)^{-1}$. Since Φ^* is an involution, this consistently confirms that $\varphi = -\Phi^*$.

Applying the local action obtained in the lemma 3.11, combined with the involution's behavior on the transcendental lattice, we can deduce the global anti-symplectic property of this transformation across the entire moduli space.

Proposition 3.15. *Let $g \geq 2$ even. Let (X, H) be a K3 surface and $\mathcal{S}$ the stable bundle of Mukai vector $(2, H, g/2)$. Let $v = (v_0, v_1 H, (g-1)v_1 - \frac{v_0}{2}(\frac{g}{2} - 1))$. Assuming $U(v)$ is a non-empty Zariski open set and* $\dim M(v) \geq 4$*, the involution $\Phi_{\mathcal{S},1}$ on $M(v)$ is anti-symplectic.*

Proof. Let $T_{M(v)}$ be the transcendental lattice of $M(v)$. We will show that $\Phi^*_{\mathcal{S},1}|_{T_{M(v)}} = -id_{T_{M(v)}}$, which immediately implies the anti-symplectic since the holomorphic symplectic form ω is a generator of $T_{M(v)} \otimes \mathbb{C}$.

Consider the exact sequence of cohomology:

$$H^2(M(v), U(v), \mathbb{Z}) \xrightarrow{f} H^2(M(v), \mathbb{Z}) \xrightarrow{\iota_v^*} H^2(U(v), \mathbb{Z})$$

According to [?, Section 11.1], $\mathrm{Im}(f) \subset \mathrm{Pic}(M(v))$. Consequently, the restriction map ι_v^* induces an injection of the transcendental lattice $T_{M(v)} \hookrightarrow H^2(U(v), \mathbb{Z})$.

Furthermore, any element in $T_{M(v)}$ can be represented as $\theta(0, x, 0)$ for some $x \in T_X$, because the map θ respects the Hodge structure. By definition, the vector $d_v = \theta(2, H, \frac{g}{2} - 1)$ lies in $Pic(M(v))$.

Since the Picard lattice is orthogonal to the transcendental lattice with respect to the Beauville-Bogomolov form, d_v is strictly orthogonal to $T_{M(v)}$, it means $B_{M(v)}(d_v, \alpha) = 0$, so we obtain $\Phi^*_{\mathcal{S},1}(\alpha) = -\alpha$. Therefore, the holomorphic symplectic form maps to $-\omega$, proving that the involution $\Phi_{\mathcal{S},1}$ on $M(v)$ is anti-symplectic. □

This sequence of results ultimately verifies the symplectic geometric features of all the new examples constructed in Theorem 4.4, establishing their status as genuine anti-symplectic symmetries.

Corollary 3.16. *The involutions defined in Theorem 3.7 are anti-symplectic.*

Now we are ready to construct the Beri-Manivel involution.

Example 3.17. For $g = 10$, we consider the moduli space $M(1, H, 7) \cong X^{[3]}$. The involution is constructed using the rank 2 spherical bundle $\mathcal{S}$ with $v(\mathcal{S}) = (2, H, 5)$, and the functor is $\Phi(\mathcal{E}) = R\mathcal{H}om(T_{\mathcal{S}}(\mathcal{E}), \mathcal{O}_X)[1] \otimes \mathcal{O}_X(H)$. Let us rigorously trace how this functor maps $\mathcal{E} := \mathcal{I}_Z(H)$ to $\mathcal{I}_{Z'}(H)$.

By direct computation we know $\langle v(\mathcal{S}), v(\mathcal{E})\rangle = -1$, then since $\mathcal{S}$ is spherical and $\mathcal{E}$ is stable, $\mathrm{Ext}^1(\mathcal{S}, \mathcal{E}) = \mathrm{Ext}^2(\mathcal{S}, \mathcal{E}) = 0$, $\dim \mathrm{Hom}(\mathcal{S}, \mathcal{E}) = 1$, that guarantees a *unique* non-zero evaluation morphism $e : \mathcal{S} \to \mathcal{I}_Z(H)$, let $\mathcal{I}_W(H)$ be its image. We obtain the exact sequence:

$$0 \to \ker(e) \to \mathcal{S} \xrightarrow{e} \mathcal{I}_W(H) \to 0 \tag{6}$$

Comparing the first Chern classes, $c_1(\ker(e)) = c_1(\mathcal{S}) - c_1(\mathcal{I}_W(H)) = H - H = 0$. For a stable bundle, a rank-1 subsheaf with trivial c_1 must be $\mathcal{O}_X$, so we get $\mathrm{length}(W) = c_2(\mathcal{S}) = 6$.

Geometric Meaning: The injection $\mathcal{O}_X \hookrightarrow \mathcal{S}$ corresponds to choosing a global section of $\mathcal{S}$. Since $H^0(\mathcal{S}) \cong \mathbb{C}^7$, specifying a section defines a 6-dimensional subspace $V_6 \subset \mathbb{C}^7$. The zero locus of this section, which is exactly W, mathematically represents the intersection $X \cap \mathrm{Gr}(2, V_6)$! Since e factors through $\mathcal{I}_Z(H)$, we have $\mathcal{I}_W(H) \subset \mathcal{I}_Z(H)$, meaning $Z \subset W$. Thus, $W = Z \cup Z'$, where Z' is the length-3 residual.

Let $e : \mathcal{S} \to \mathcal{E}$ and cone $C = T_{\mathcal{S}}(\mathcal{E})$ fits into the distinguished triangle $\mathcal{S} \to \mathcal{E} \to C$. Taking the long exact sequence of cohomology sheaves, the complex C fits into the triangle:

$$\mathcal{O}_X[1] \to C \to \mathcal{O}_{Z'} \xrightarrow{+1} \tag{7}$$

by taking duality, the dualized triangle becomes

$$\mathcal{O}_{Z'}[-2] \to C^\vee \to \mathcal{O}_X[-1] \xrightarrow{+1}$$

using $R\mathcal{H}om(\mathcal{O}_{Z'}, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{O}_{Z'}, \mathcal{O}_X)[-2] \cong \mathcal{O}_{Z'}[-2]$ Shifting by [1] and rotating, we obtain

$$\mathcal{O}_{Z'}[-1] \to \Phi(\mathcal{E}) \to \mathcal{O}_X(H) \xrightarrow{+1}$$

taking long exact sequence of cohomology, and knowing that $H^1(\Phi) = 0$, that means

$$0 \to \Phi(\mathcal{E}) \to \mathcal{O}_X(H) \to \mathcal{O}_{Z'} \to 0$$

Then the results naturally follow from the exact sequence

$$0 \to \mathcal{I}_{Z'}(H) \to \mathcal{O}_X(H) \to \mathcal{O}_{Z'} \to 0$$

3.5 Action of $\Phi_{\mathcal{S},1}$ on cohomology

To describe these symmetries more precisely, we need to delve into their specific behavior on $H^2(M(v),\mathbb{Z})$, particularly their impact on the Picard lattice. To achieve this, we first provide a standard basis and the discriminant for $\mathrm{Pic}(M(v))$.

Lemma 3.18. *Let X be a K3 surface with $\mathrm{Pic}(X)=\mathbb{Z}\cdot H$. Let $v=(v_0,v_1H,v_2)$ with $v_2=(g-1)v_1-\frac{v_0}{2}(\frac{g}{2}-1)$. Set $\delta:=v_0\wedge v_2$, $d_v=\theta(2,1,\frac{g}{2}-1)$ and $f_v:=\frac{1}{\delta}\theta(v_0,0,-v_2)$. Then (d_v,f_v) gives a basis of $\mathrm{Pic}(M(v))$ with the Gram matrix:*

$$\begin{pmatrix} 2 & \frac{2v_2}{\delta}-(\frac{g}{2}-1)\frac{v_0}{\delta} \\ \frac{2v_2}{\delta}-(\frac{g}{2}-1)\frac{v_0}{\delta} & 2\frac{v_0v_2}{\delta^2} \end{pmatrix}$$

In particular, the discriminant is $\mathrm{disc}(\mathrm{Pic}(M(v)))=-\frac{4(g-1)}{\delta^2}(n-1)$, where $2n=\dim M(v)$.

Equipped with the basis and the discriminant of the lattice, we can use lattice theory to completely classify the invariant sublattice of the involution, which tightly depends on the dimension of the moduli space and the arithmetic of the genus.

Proposition 3.19. *Let $g\equiv 2 \pmod 4$ and $v=(2k+1,(k+1)H,v_2)$. Let $2n:=\dim M(v)\geq 4$. The action of $\Phi_{\mathcal{S},1}$ on cohomology is described as follows:*

- *For $n>2$, if $H^2(M(v),\mathbb{Z})^{\Phi_{\mathcal{S},1}}\simeq(2(n-1))$, then $\delta|\gcd(n-1,g-1)$ and $\frac{-\delta^2}{g-1}$ is a square in $\mathbb{Z}/(n-1)\mathbb{Z}$.*
- *For even $n>2$, if $H^2(M(v),\mathbb{Z})^{\Phi_{\mathcal{S},1}}\simeq(n-1)$, then $\delta|\gcd(n-1,g-1)$ and $\frac{-\delta^2}{2(g-1)}$ is a square in $\mathbb{Z}/(n-1)\mathbb{Z}$.*
- *Otherwise, the invariant part is $H^2(M(v),\mathbb{Z})^{\Phi_{\mathcal{S},1}}\simeq(2)$.*

Remark 3.20. In the original paper, the authors wrongly deduce $\delta=1$ from the condition $\delta^2|(g-1)$. However, strictly speaking, δ only needs to divide $\gcd(n-1,g-1)$, and it does not necessarily have to be 1.

Remark 3.21. Lemma 3.18 provides the exact Gram matrix and discriminant of $\mathrm{Pic}(M(v))$. This translates our geometric search for the anti-symplectic invariant sublattice $H^2(M(v),\mathbb{Z})^{\Phi_{\mathcal{S},1}}$ into this rank 2 lattice $\mathrm{Pic}(M(v))$ and then we can using classification theorem according to [CCC21, Proposition 1.6].

Remark 3.22. The proof tests the allowable discriminant groups for $K3^{[n]}$-type manifolds. Rank-2 invariant sublattices are eliminated because they violate the dimension bound $\dim M(v)\geq 4$. This restricts the invariant part to rank 1, naturally transforming the intersection form into the quadratic residue conditions.

Finally, we must emphasize why this entire lattice classification is possible in the first place.

Remark 3.23. The assumption $\mathrm{Pic}(X)=\mathbb{Z}H$ ensures the algebraic Mukai lattice has rank 3, forcing $\mathrm{Pic}(M(v))\simeq v^{\perp}$ to have rank exactly 2. If the Picard rank were higher, $\mathrm{Pic}(M(v))$ would exceed rank 2, completely invalidating the 2×2 Gram matrix classification.

4 Applications to Cubic 4 folds

4.1 Basic results on (anti)-symplectic involutions on cubic 4 folds

According to Marquand (2022), involutions on a cubic fourfold X fall into three distinct types:

Theorem 4.1. *Let X be a general cubic fourfold with ϕ_i an involution of X fixing a linear subspace of $\mathbb{P}^5$ of codimension i. Then either*

- *$i = 1, \phi_1$ is anti-symplectic and $A(X)_{prim} \cong E_6(2), T(X) \cong U^2 \oplus D_4^3$. The algebraic lattice is spanned by classes of planes contained in X;*
- *$i = 2, \phi_2$ is symplectic and $A(X)_{prim} \cong E_8(2), T(X) \cong A_2 \oplus U^2 \oplus E_8(2)$. The algebraic lattice is spanned by classes of cubic scrolls contained in X;*
- *$i = 3, \phi_3$ is anti-symplectic and*

$$A(X)_{prim} \cong M, T(X) \cong U \oplus A_1 \oplus A_1(-1) \oplus E_8(2)$$

where $A(X)_{prim} = H^{2,2}(X,\mathbb{C}) \cap H^4_{prim}(X,\mathbb{Z})$ and $T(X) := A(X)^{\perp}_{prim} \subset H^4_{prim}(X,\mathbb{Z})$ orthogonal complement with respect to Mukai pairing. The algebraic lattice contains an index 2 sublattice spanned by classes of planes contained in X. Here M is the unique rank 10 even lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24 \rangle$.

Now we focus on the generators of $A(X)_{prim}$

Theorem 4.2.

- ***Case ϕ_1:*** *The primitive algebraic lattice $A(X)_{prim} \cong E_6(2)$ is generated by the differences of classes of planes $[\Pi_i] - [\Pi_j]$, where Π_i are the 27 planes passing through the Eckardt point.*
- ***Case ϕ_2:*** *The primitive algebraic lattice $A(X)_{prim} \cong E_8(2)$ is generated by the classes $\alpha_i := [T_i] - \eta_X$, where $[T_i]$ are the classes of cubic scrolls contained in X, and η_X is the square of the hyperplane class.*
- ***Case ϕ_3:*** *The primitive algebraic lattice $A(X)_{prim} \cong M$ is generated by the classes $\{x, \alpha_1, \dots, \alpha_9\}$, where:*
  - $\alpha_i = [F_i] - [F_{i+1}]$ *for* $1 \le i \le 8$,
  - $\alpha_9 = [P] + [F_8] + [F_9] - \eta_X$,
  - $x = \frac{\alpha_1+\alpha_3+\alpha_5+\alpha_7+[F_9]-[P]}{2}$,

  with P being the point-wise fixed plane, and F_i being invariant planes contained in X.

Remark 4.3. *For construction of generators of ϕ_1:*

- Let X be a cubic fourfold with an anti-symplectic involution ϕ_1, which is geometrically equivalent to X possessing an Eckardt point p.
- The fixed locus of the involution ϕ_1 consists of this Eckardt point p and a complementary hyperplane $\mathbb{P}^4$.
- Geometrically, X contains a cone over a smooth cubic surface S, with the vertex of the cone being the Eckardt point p.
- The 27 lines on this smooth cubic surface S sweep out exactly 27 planes Π_i $(1 \le i \le 27)$ in X, all of which intersect at the vertex p.
- The primitive algebraic lattice $A(X)_{\text{prim}} \cong E_6(2)$ is generated by the difference classes $[\Pi_i] - [\Pi_j]$ of these 27 planes.

For construction of generators of ϕ_2:

- Let X be a general cubic fourfold with a symplectic involution ϕ_2. The fixed locus of ϕ_2 consists of a line l and a disjoint cubic surface $S = \Pi \cap X$, where $\Pi \cong \mathbb{P}^3$ is the complementary fixed subspace.

- The linear projection from the fixed line l to the subspace Π induces a conic fibration $\pi : Bl_l X \to \Pi$.
- The discriminant locus of this fibration is the union of the fixed cubic surface S and a quadric surface Q. Their intersection $C = S \cap Q$ is a genus 4 space canonical curve.
- This canonical curve C admits 120 tritangent planes $\Gamma_i \subset \Pi$ ($1 \leq i \leq 120$).
- For each tritangent plane Γ_i, the hyperplane $H_i = \operatorname{span}\{l, \Gamma_i\}$ sections X into a cubic threefold $Y_i = X \cap H_i$ possessing exactly three pairs of nodes interchanged by the involution.
- By the geometry of nodal cubic threefolds, each Y_i contains a pair of families of cubic scrolls T_i and T_i' whose cohomology classes satisfy $[T_i] + [T_i'] = 2\eta_X$.
- The primitive lattice $A(X)_{\text{prim}} \cong E_8(2)$ is generated by the 120 classes $\alpha_i := [T_i] - \eta_X$.

4.2 The Rigidity of $\mathcal{D}^b(X)$ vs. the Flexibility of $\mathcal{A}_X$

A fundamental obstacle in categorical reconstruction is the **Bondal-Orlov Reconstruction Theorem**. Since X is a Fano variety ($K_X = \mathcal{O}_X(-3)$), the autoequivalence group is rigid:

$$\operatorname{Aut}(\mathcal{D}^b(X)) \cong \operatorname{Aut}(X) \ltimes (\operatorname{Pic}(X) \times \mathbb{Z}) \tag{8}$$

This implies that no spherical objects (and thus no spherical twists) can exist in the global category $\mathcal{D}^b(X)$, explaining why anti-symplectic involutions (like ϕ_1) cannot be realized via simple categorical surgery.

However, the Kuznetsov component $\mathcal{A}_X$, defined by the semi-orthogonal decomposition:

$$\mathcal{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle \tag{9}$$

is a non-commutative K3 surface with Serre functor $S_{\mathcal{A}_X} \cong [2]$. This allows for the existence of spherical objects and their associated twists.

4.3 Construction of the 240 Spherical Objects

We define 240 objects E_i in $\mathcal{A}_X$ associated with the 120 tritangent planes H_i of a genus 4 degree 6 curve $C \subset \mathbb{P}^3$.

1. Let M_i be the simple modules such that $\mathcal{E}nd_{\mathcal{O}_{H_i}}(M_i) \cong \mathcal{B}_0|_{H_i}$.
2. We define $E_i := \operatorname{pr}_{\mathcal{A}_X}(Rp_*(q^* M_i \otimes \mathcal{E}))$, where $\mathcal{E}$ is the universal spinor bundle.
3. We prove that $\operatorname{Ext}^\bullet_{\mathcal{A}_X}(E_i, E_i) \cong \mathbb{C} \oplus \mathbb{C}[-2]$ and $S_{\mathcal{A}_X}(E_i) \cong E_i[2]$, confirming E_i are 2-dimensional spherical objects.

Here we construct the spherical object E_i in detail.

Definition 4.4.

- Let X be a cubic fourfold containing a plane P. The blow-up of X along P gives a space $\tilde{X} = Bl_P(X)$. This space admits two natural morphisms: the blow-down map $p : \tilde{X} \to X$, and the quadric fibration $q : \tilde{X} \to \mathbb{P}^2$ induced by the linear projection from P.
- Associated to the quadric fibration $q : \tilde{X} \to \mathbb{P}^2$, there exists a sheaf of even Clifford algebras $\mathcal{B}_0$ on the base space $\mathbb{P}^2$.

- Let $H_i \subset \mathbb{P}^2$ be specific loci (e.g., irreducible components of the discriminant curve) over which the Brauer class of $\mathcal{B}_0$ is trivial. Thus, the restricted Azumaya algebra $\mathcal{B}_0|_{H_i}$ splits and is isomorphic to an endomorphism algebra. We define M_i as the splitting bundle on H_i, meaning M_i is a simple module satisfying $\mathcal{E}nd_{\mathcal{O}_{H_i}}(M_i) \cong \mathcal{B}_0|_{H_i}$.
- Let $\mathcal{E}$ be the universal spinor bundle on the quadric fibration $\tilde{X}$.
- We pull back the module M_i to $\tilde{X}$ via q^*, tensor it with the universal spinor bundle $\mathcal{E}$, and then push the resulting complex forward to X via the derived functor Rp_*. This yields an object in $D^b(X)$.
- Finally, we apply the projection functor

$$\mathrm{pr}_{\mathcal{A}_X} =: \mathbb{L}_{\mathcal{O}_Y}\mathbb{L}_{\mathcal{O}_Y(H)}\mathbb{L}_{\mathcal{O}_Y(2H)} : D^b(X) \to \mathcal{A}_X$$

  to obtain the object $E_i \in \mathcal{A}_X$.

We construct a categorical autoequivalence $\Phi \in \mathrm{Aut}(\mathcal{A}_X)$ as:

$$\Phi = \mathbb{D} \circ \prod_{i=1}^{120} T_{E_i} \tag{10}$$

The composition of 120 twists induces the central element $-\mathrm{id}$ in the $W(E_8)$ Weyl group on the algebraic lattice, this action on the Mukai lattice $\tilde{H}(\mathcal{A}_X, \mathbb{Z})$ uniquely corresponds to the symplectic involution ϕ_2.

4.4 Descent to Geometry via the Moduli Space

Definition 4.5. For any line $l \subset X$, let $\mathcal{I}_l$ be the ideal sheaf of l in X. The object $P_l \in \mathcal{A}_X$ is defined as the projection of $\mathcal{I}_l$ into the Kuznetsov component, denoted as $P_l := \mathrm{pr}_{\mathcal{A}_X}(\mathcal{I}_l)$. Equivalently, P_l is uniquely determined by the following distinguished triangle in $D^b(X)$:

$$\mathcal{O}_X(-H)[1] \longrightarrow P_l \longrightarrow \mathcal{I}_l \longrightarrow \mathcal{O}_X(-H)[2]$$

It has some basic properties:

1. *Mukai Vector:* In the numerical Grothendieck group $K(\mathcal{A}_X)$, the Mukai vector of P_l is exactly $v(P_l) = \lambda_1 + \lambda_2$, where λ_1 and λ_2 span the standard A_2 lattice in $\mathcal{A}_X$.
2. *Bridgeland Stability:* For a suitably constructed Bridgeland stability condition σ on $\mathcal{A}_X$, the object P_l is always σ-stable.
3. *Moduli Space Representation:* The moduli space of σ-stable objects in $\mathcal{A}_X$ with Mukai vector $\lambda_1 + \lambda_2$, denoted $M_\sigma(\lambda_1 + \lambda_2)$, is canonically isomorphic to the Fano variety of lines $F(X)$.

To retrieve the geometric morphism $\phi_2 \in \mathrm{Aut}(X)$, we use the Fano variety of lines $F(X)$.

Definition 4.6. Let $\Phi : \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$ be an autoequivalence of the Kuznetsov component. The induced geometric mapping $f_\Phi : F(X) \to F(X)$, given by $l \mapsto l'$, is constructed through the following steps:

- For a given line $l \in F(X)$, consider its associated projection object $P_l := \mathrm{pr}_{\mathcal{A}_X}(\mathcal{I}_l) \in \mathcal{A}_X$. The autoequivalence Φ acts on this object to produce a new object $\Phi(P_l) \in \mathcal{A}_X$.
- Since the autoequivalence Φ is assumed to preserve the stability condition σ (**Not sure!**) and the numerical class, the resulting object $\Phi(P_l)$ remains a strictly σ-stable object with the identical Mukai vector $v(\Phi(P_l)) = \lambda_1 + \lambda_2$.

- Since $M_\sigma(\lambda_1 + \lambda_2) \cong F(X)$. Under this identification, the σ-stable object $\Phi(P_l)$ must correspond to a unique closed point in the Fano variety of lines.
- Consequently, there exists a unique line $l' \subset X$ such that there is an isomorphism in the derived category: $\Phi(P_l) \cong P_{l'}$. This uniquely defines a map $f_\Phi : F(X) \to F(X)$ where $f_\Phi(l) = l'$.
- The mapping $f_\Phi \in \mathrm{Aut}(F(X)) \cong \mathrm{Aut}(X)$, by Voisin's theorem for cubic fourfolds, the map f_Φ uniquely determines a geometric automorphism $\phi \in \mathrm{Aut}(X)$ such that $l' = \phi(l)$.

We need to check it is exactly ϕ_2.

References

[BO97] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, arXiv preprint alg-geom/9712029 (1997).

[CCC21] C. Camere, A. Cattaneo, and A. Cattaneo, *Non-symplectic involutions on manifolds of $K3^{[n]}$-type*, Nagoya Math. J., 243:278–302 (2021).

[FMM25] D. Faenzi, G. Menet, and Y. Prieto-Montañez, *Anti-symplectic involutions on moduli spaces of sheaves on K3 surfaces via auto-equivalences*, arXiv preprint arXiv:2409.12668 (2025).

[Kuz08] A. Kuznetsov, *Derived categories of cubic fourfolds*, arXiv preprint arXiv:0808.3351 (2008).

[LPZ20] C. Li, L. Pertusi, and X. Zhao, *Twisted cubics on cubic fourfolds and stability conditions*, arXiv preprint arXiv:1802.01134v2 (2020).

[MR20] G. Menet and U. Rieß, *On the Kähler cone of irreducible symplectic orbifolds*, arXiv preprint arXiv:2009.04873 (2020).

[Voi20] C. Voisin, *Schiffer variations and the generic Torelli theorem for hypersurfaces*, arXiv preprint arXiv:2004.09310 (2020).