Survey on Allcock's Bimonster Conjecture

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Contents

1 Introduction

The monstrous proposal that the orbifold fundamental group $\pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$ modulo the square of all meridians is isomorphic to the bimonster group is an interesting conjecture. So far, it has been proved that this orbifold fundamental group is either the bimonster or $\mathbb{Z}/2$, where the bimonster is the group $(M \times M) \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts by exchange factors of the monster group M. It does not seem interesting, since it's just a result of finding the bimonster as a orbifold fundamental group of a particular perhaps even complicated figure.

However, there's more under this problem. There are many results related to such ball quotient. For example, the moduli space of genus 3 curves is birational to an arithmetic quotient of a 6-dimensional ball, and the moduli space of genus 4 curves is birational to an arithmetic quotient of a 9-dimensional ball. Moreover, the moduli space of pointed genus 2(resp. 3) curves also have a ball quotient structure.

Another kind of such example is about the Deligne-Mostow 9-ball. It is of the similar construction of the 13-dimensional ball quotient in the proposal, a ball taking out all mirrors of triflections of roots in an Eisenstein lattice, then quotient it by the automorphism group of the lattice. $(\mathbb{B}^9_{\text{DM}} - \mathcal{H}_{\text{DM}})/P\Gamma_{\text{DM}}$ is isomorphic to the moduli spaces of unordered 12-tuples of distinct points. It should be notice that $\mathbb{C}P^{12}/S_{12}$ is isomorphic to $\mathbb{CP}^{12}$, thus the orbifold is a suborbifold of $\mathbb{C}P^{12}$. And as we know, the orbifold fundamental group of the moduli spaces of unordered 12-tuples is the 12-braid group.

Thus we can guess, if the conjecture holds, that the orbifold fundamental group of $\pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$ modulo the square of all meridians is isomorphic to the bimonster, we can furthermore try to build a connection between the orbifold and a moduli space of some curves. Then consider its orbifold cover, we will get an action of the bimonster on a family of some objects. This should be very interesting.

The origin of this conjecture is a coincidence of diagrams. It is conjectured that the bimonster can be presented as the quotient of the Coxeter grup with diagram Y_{555} by a single extra relation. (This part will be introduced briefly).

There are also some relevant evidence that this should holds other than the direct proof which will be mentioned in this paper.

2 Basic Settings

This part follows the notation of [AB23].

2.1 Eisenstein lattices

Let $\omega = e^{\frac{2\pi i}{3}}$ and $\theta = \omega - \bar{\omega} = \sqrt{-3}$. Let $\mathcal{E}$ by the ring $\mathbb{Z}[\omega]$ of Eisenstein integers. An Eisenstein lattice is a free $\mathcal{E}$-module K equipped with a hermitian form $\langle \mid \rangle$ $K \times K \to \mathbb{Q}(\omega)$. We usually see K as a submodule in $K \otimes \mathbb{C}$, i.e. $K \otimes_{\mathcal{E}} \mathbb{C}$. When K is Lorentzian, we write $\mathbb{B}(K)$ for $\mathbb{B}(K \otimes \mathbb{C})$, which consists of all vectors with zero length. If K is non-degenerate, then its dual lattice is defined as $K^* = \{x \in K \otimes \mathbb{C} \mid \langle x|k \rangle \in \mathcal{E} \text{ for all } k \in K\}$.

2.2 The lattice L

The lattice L is a central object in this problem. It is the associated Eisenstein lattice to the graph $\mathbb{P}_2 F_3$. We explain this meaning in the following.

Given a directed graph Δ without self-loops or multiple edges, we consider the free $\mathcal{E}$-module generated by the vertices of Δ. The basis vectors are denoted by e_α indexed by $\alpha \in \Delta$. We associated a hermitian form to this lattice:

$$\langle e_\alpha | e_\beta \rangle = \begin{cases} 3, & \text{if } \alpha = \beta \\ \theta, & \text{if there is an edge from } \beta \text{ to } \alpha \\ 0, & \text{if there is no edge between } \beta \text{ and } \alpha \end{cases} \tag{1}$$

Now, we consider the 13 points and 13 lines in $P_2\mathbb{F}_3$, we label the points by $p_1, \cdots, p_{13}$ and the lines by $l_1, \cdots, l_{13}$, s.t. the points on l_j are $p_i, p_{i+1}, p_{i+3}, p_{i+9}$ (subscripts mod 13). We draw a graph, where vertices are points and lines, we direct an edge from a line to any point lies on it, and there is no edges between lines or between points. Later, we will refer to this graph as graph $P_2\mathbb{F}_3$.

Then the corresponding Eisenstein lattice is 26-dimensional, and the hermitian form on it has radical with dimension 12. We consider the lattice quotient the radical, it is a 14-dimensional lattice with a non-degenerate hermitian form with signature $(13, 1)$. We denote this lattice by L in this paper. It is the unique $\mathcal{E}$-lattice of signature $(13, 1)$ satisfying $L = \theta L^*$.

There is a particular model for this lattice. There's a suitable choice of coordinates of $\mathbb{C}^{13,1}$ with standard Lorentzian metric $\langle x|y\rangle = -x_0\bar{y_0}+x_1\bar{y_1}+\cdots+x_{13}\bar{y_{13}}$, and $p_1 = (0 : \theta, 0, \cdots, 0)$, $l_1 = (1 : 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$, and rightward cyclic permutation of the last 13 coordinates increases subscripts by 1.

We write Γ for the isometry group of L. Obviously it contains the group $L_3(3) := \mathrm{PGL}(\mathbb{F}_3)$, permuting the points and lines of $P^2\mathbb{F}_3$ in the natural way. And also, there is an additional symmetry, which maps p_i to l_{10i} (subscripts modulo 13), i.e. mapping the point $(a : b : c)$ to the line $ax + by + cz = 0$. Then they generate a subgroup of Γ whose image in $P\Gamma$ is $L_3(3) \rtimes 2$.

2.3 Roots in L, mirrors and the hyperplane arrangement $\mathcal{H}$

A root of L is a lattice vector of norm 3. Given a root $s \in L$, it admits a triflection, an element in Γ, mapping

$$x \mapsto x - (1-\omega)\frac{\langle x|s\rangle}{s^2}.$$

We should remark that Γ is generated by the triflections corresponding to the 26 point- and line-roots.

The hyperplane $\mathbb{B}(s^\perp)$ is called the mirror of s. Our goal is to introduce the results on the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$.

2.4 The base point for the orbifold $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$

A convenient basepoint for the orbifold fundamental group is the midpoint τ of the geodesic segment joining p_∞ and l_∞ where, p_∞ resp. l_∞ is of norm -3 orthogonal to $p_1, \cdots, p_{13}$ resp. $l_1, \cdots, l_{13}$. The explicit coordinates are as follow:

$$\tau = l_\infty + ip_\infty = (4+\sqrt{3}; 1^{13})$$

, and $p_\infty = (\bar{\theta}; 0, \cdots, 0)$ resp. $l_\infty = (4; 1, \cdots, 1)$. Then τ is of norm $-6-8\sqrt{3}$, and it's the unique fixed point of $L_3(3) \rtimes 2 \subset P\Gamma$ and the latter one is exactly the local group of this point. The mirrors closest to this points are just the 26 point- and line- mirrors. (See [Bas06, prop 1.2], where τ was called $\bar{\rho}$).

2.5 The Deligne-Mostow lattice

The lattice L_{DM} is both a sublattice of L and a lower-dimensional anologue of L. We first gives its construction. Consider the 12-gon, i.e. the $\tilde{A}_{11}$ Dynkin diagram (a 12-gon), with its edges' orientations alternating. Then the associated Eisenstein lattice is of rank 12 with a radical of dimension 2. We define L_{DM} as the quotient of the lattice by the radical. It has signature $(9, 1)$ and satisfies $\theta L^*_{\mathrm{DM}} = L_{\mathrm{DM}}$. We define $\mathbb{B}^9_{\mathrm{DM}}$ as $\mathbb{B}(L_{\mathrm{DM}})$, and Γ_{DM} as the isometry group of L_{DM}.

By construction, since D_{24} permutes the 12 vertices of $\tilde{A}_{11}$, and maps edges to edges, we have $P\Gamma_{DM}$ contains D_{24}.

Since $P^2\mathbb{F}_3$ contains many 12-gon, L_{DM} appears as a sublattice of L, and there are many ways to embed it into L. Because $L_{\mathrm{DM}} = \theta L^*_{\mathrm{DM}}$ and all inner products in L are divisible by θ, we can show that L_{DM} is a summand. Therefore, Γ_{DM} is a subgroup of Γ. In the following chapter, we take a particular choice of 12-gon. For example, the 12-gon

$$p_6, l_{10}, p_{13}, l_4, p_7, l_{11}, p_{12}, l_9, p_9, l_8, p_8, l_5.$$

This will be the model for the subsection change of basepoints 4.1. The group $(L_3(3) \rtimes 2)$ acts on the graph $P^2\mathbb{F}_3$, and the subgroup which exactly fixed the 12-gon is just the dihedral group D_{24}.

The projection of $\tau \in \mathbb{B}_{13}$ to $\mathbb{B}^9_{\mathrm{DM}}$ is denoted by ρ. Later, we will introduce that when moving the basepoint $\sigma \neq \rho$ along the segment $\bar{\rho\tau}$, the orbifold fundamental group $\pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \sigma)$ does not change under isomorhpism. This will be important when we try to build a connection between the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ and $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$.

2.6 Orbifold fundamental groups

Let A be a group acting properly discontinuously on a path connected manifold X. Choose a base point $b \in X$. Then we define the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/A, b)$ as the following set of equivalence classes of pairs (γ, g), where $g \in A$ and γ is a path in X from b to gb. One such pair is equivalent to another one (γ', g') if $g = g'$ and γ, γ' are homotopic in X, rel endpoints. The group operation is

$$(\gamma, g) \cdot (\gamma', g') = (\gamma \text{ followed by } g \circ \gamma', gg').$$

Inversion in $\pi_1^{\text{orb}}(X/A, b)$ is given by $(\gamma, g)^{-1} = (g^{-1} \circ \text{reverse}(\gamma), g^{-1})$. And, there is a natural projection map from $\pi_1^{\text{orb}}(X/A, b)$ to A by mapping (γ, g) to g. Easy to verify this is s surjective group homomorphism since X is path-connected. The kernel is obviously $\pi_1(X, b)$, yielding the exact sequence

$$1 \to \pi_1(X, b) \to \pi_1^{\text{orb}}(X/A, b) \to A \to 1.$$

The local group at b means that the set of $(\gamma, g) \in \pi_1^{\text{orb}}(X/A, b)$ for which γ is homotopic to the constant path at b. It should be notice that it's the same as the A-stabilizer of b. In this paper, we just need to know this simplified version of orbifold fundamental groups since all orbifold appeared in this paper are just a quotient of a manifold by a group action, not the kind glued of many such pieces.

2.7 Meridians

Meridians are distinguished elements of $\pi_1^{\text{orb}}((\mathbb{B}_{13} - \mathcal{H})/P\Gamma, b)$ or $\pi_1^{\text{orb}}((\mathbb{B}^9_{\text{DM}} - \mathcal{H}_{\text{DM}})/P\Gamma_{\text{DM}}, b)$. If s is a root in the lattice L or L_{DM}, and S is the triflection of s, then the corresponding meridian $M_{b,s}$ is defined as $(\mu_{b,s}, S)$ where $\mu_{b,s}$ is a path from b to Sb as follow: let p be the projection of b into the mirror $s^\perp$. In this paper, $\bar{bp}$ never meets any other mirror. Now, we choose a ball

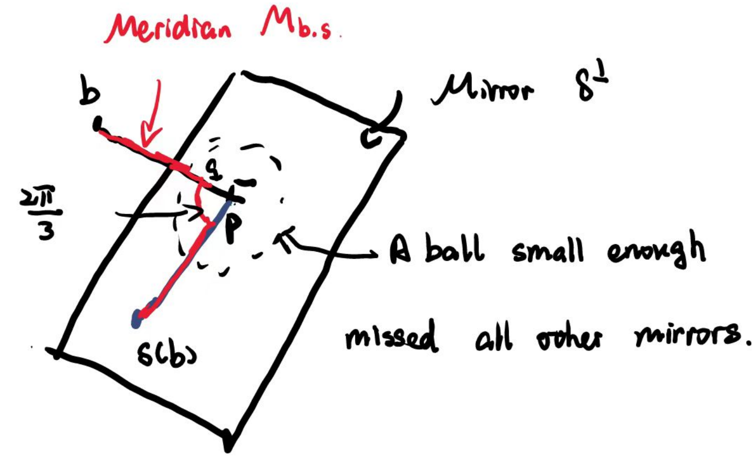

Figure 1:

around p small enough such that it miss all other mirrors, and we take q to be a point of $\bar{bp} - \{p\}$ in this ball. Then $\mu_{b,s}$ is the geodesic $\bar{bq}$, followed by a positive circular arc in $\mathbb{B}^1(b, q)$ of angle $\frac{2\pi}{3}$ centered at p, followed by $S(\bar{qb})$.

3 The Monstrous Proposal

Conjecture 1 (D.Allcock, 2007). *The quotient of G by the normal subgroup generated by the squares of the meridians is the bimonster, i.e., the semidirect product $M \times M : 2$, where M is the bimonster group and $\mathbb{Z}/2$ acts by exchanging factors.*

In 2007, Allcock give this conjecture in [All07]. In [AB23, Theorem 1.1], they proved the following result, which is a big step toward this goal:

Theorem 1 (Allcock-Basak, 2023). *The complex hyperbolic braid group $G = \pi_1^{orb}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$, modulo the subgroup S generated by the squares of all meridians, is isomorphic to either the bimonster or $\mathbb{Z}/2$.*

And this is just one step away from proving the monstrous proposal, that is ruling out the case $G/S \cong \mathbb{Z}/2$.

We do not actually determine a presentation for G, but we do know that it is the quotient of a certain Artin group, $\mathrm{Art}(P^2\mathbb{F}^3)$, with the standard Artin generators mapping to meridians. The paper proved this theorem by finding extra relations for these meridians.

4 The Sketch of the Proof

We sketch the proof of Theorem 1 in this section.

The key to the proof is first to find a 12-gon in the graph $P^2\mathbb{F}_3$, then we can find the Deligne-Mostow Ball inside the 13-dimensional ball and the corresponding lattice embedded, hence the roots and the mirrors. Then by considering a suitable neighborhood U of $\mathbb{B}^9$ s.t.

$$(U - \mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{SW}} \to (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$$

is a fibration. We can derive the following short exact sequence from the long exact sequence of orbifold homotopy groups and the fact that $(\mathbb{B}^9_{\mathrm{DM}} -$

$\mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ is isomorphic to the braid, the moduli space of 12-unordered tuples in $\mathbb{C}P^1$ has contractible covering space:

$$0 \to \mathrm{Br}_5 \to \pi_1^{\mathrm{orb}}((U-\mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{SW}}) \to \pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}) \to 0$$

where Br_5 is just the orbifold fundamental group of the fiber. And we can determine that it can be generated by five meridians in the 13-dimensional ball quotients. Also, on the last term, we can also determine that it can be generated by 12 meridians.

Then by analyzing the exact sequence, we can get the relations of the 12 meridians inside the group $\pi_1^{\mathrm{orb}}((U-\mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{SW}})$. Furthermore, we can realizes the same relation of 12 meridians inside the group $\pi_1^{\mathrm{orb}}((\mathbb{B}^{13}-\mathcal{H})/P\Gamma)$ related to what kind of embedding of the 12-gon inside $P_2\mathbb{F}_3$ we chose. Then by varying the ways of embedding, we get a large number relations of the meridians which generate the group $\pi_1^{\mathrm{orb}}((\mathbb{B}^{13}-\mathcal{H})/P\Gamma)$. This will be enough to prove that this group modulo squares of all its meridians, is a quotient group of the bimonster. Then it can only be the bimonster itself, $\mathbb{Z}/2$ or just the trivial group. By using automorphic forms, it can be shown that this group cannot be trivial.

Now, we give a more concrete sketch of [AB23].

4.1 Change of Basepoint in $\mathbb{B}^{13}$

To relate the orbifold fundamental group of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ and $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$, we need to choose specific basepoints. We have already take the point τ as the basepoint for the orbifold $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$, and a natural choice for the basepoint of the orbifold $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})$ would be the projection of ρ to $\mathbb{B}^9_{\mathrm{DM}}$, denoted by ρ. However, such a ρ lies in $\mathcal{H}$, so it doesn't make sense to speak of $\pi_1^{\mathrm{orb}}((\mathbb{B}_{13} - \mathcal{H})/P\Gamma, \rho)$. So we need to move the basepoint in the segment $\overline{\tau\rho}$ meanwhile and tracking the change of the meridians the same time.

We fix an A_4 subdiagram of the incidence graph of $P^2\mathbb{F}_3$. We choose l_1, p_2, l_2, p_3. They are mutually orthogonal except that $\langle p_2 \mid l_1\rangle = \langle p_2 \mid l_2\rangle = \langle p_3 \mid l_2\rangle = \theta$, thus their integral span is isomorphic to the Eisenstein lattice. We call it $L_4^{(1)}$. And them the L_{DM} we introduce before will just be it orthogonal complement in L. It corresponding to nodes(they are note joined to the A_4)

$$p_6, l_{10}, p_{13}, l_4, p_7, l_{11}, p_{12}, l_9, p_9, l_8, p_8, l_5$$

in cyclic order. For convenience, we denoted them by $s_0, \cdots, s_{11}$ for them, in this order. And we write s_A, s_B, s_C, s_D for the roots l_1, p_2, l_2, p_3. It should be mentioned that exactly 40 mirrors contain $\mathbb{B}_{\mathrm{DM}^9}$, corresponding to the scalar classes of the 240 roots of $L_4^{(1)}$.

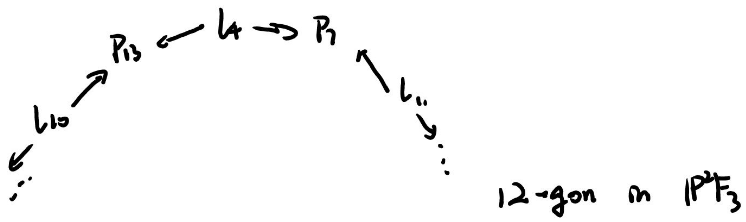

Figure 2:

Let ρ be the projection of τ to $\mathbb{B}^9_{\mathrm{DM}}$, and let σ be any point of $\overline{\tau\rho} - \{\rho\}$. We abbreviate the merisidians $M_{\tau,s_0}, \cdots, M_{\tau,s_A}, \cdots, M_{\tau,s_D}$ to $\tau_0, \cdots, \tau_{11}, \tau_A, \cdots, \tau_D$, and similarly with τ in place of τ.

Then the following theorem will be important to relate the needed orbifold fundamental groups in this paper while moving basepoints. And it track the change of meridians.

Theorem 2. *The segment $\tau\rho$ meets $\mathcal{H}$ only at ρ. For any $\sigma \in \overline{\tau\rho} - \{\rho\}$, the change-of-basepoint isomorphism*

$$\pi_1^{orb}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau) \cong \pi_1^{orb}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \sigma)$$

induced by the segment $\overline{\tau\sigma}$, identifies each meridian $\tau_0, \cdots, \tau_{11}, \tau_A, \cdots, \tau_D$ based at τ with the corresponding meridian $\sigma_0, \cdots, \sigma_{11}, \sigma_A, \cdots, \sigma_D$ based at σ.

Remark 1. *This theorem requires that image of τ_i is σ_i, i.e. (s_i, M_{τ,s_i}) connecting with the segment $\overline{\tau\sigma}$ is homotopic to (s_i, M_{σ,s_i}), i.e. the path M_{τ,s_i} with two vertices linked by the paths $\overline{\tau\sigma}$ and $s_i(\overline{\tau\sigma})$ is homotopic to the path M_{σ,s_i}. $(i = 1, \cdots, 11, A, B, C, D)$.*

This directly holds from the next lemma. The proof is a series of computation, and they will not be put in this survey.

Lemma 1. (1) *Fix $j = 0, \cdots, 11$, and let Q be the totally real quadrilateral with vertices τ, ρ, and the projections τ', ρ' of these points to $s_j^\perp$. Then Q meets $s_j^\perp$ in $\overline{\tau'\rho'}$, meets the 40 mirrors containing $\mathbb{B}^9_{\mathrm{DM}}$ in $\overline{\rho\rho'}$, and misses all other mirrors.*

(2) *Fix $j = A, \cdots, D$, and let T be the totally real triangle with vertices τ, ρ and the projection τ' of τ to $s_j^\perp$. Then T meets $s_j^\perp$ in $\overline{\tau'\rho}$, meets the other 39 mirrors containing $\mathbb{B}^p_{\mathrm{DM}}$ at ρ only, and misses all other mirrors.*

The next lemma follows from the proof of lemma 1.

Lemma 2. *The components of $\mathcal{H}$ that come nearest ρ, other than those that pass through it, are $s_0^\perp, \cdots, s_{11}^\perp$.*

The fact leads to an important result in the later section, that the local group of ρ in the orbifold $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ i.e. the $P\Gamma_{\mathrm{DM}}$-stabilizer permutes these 12 mirrors, thus the stabilizer is a subgroup of D_{24}. From section 2.5, we can see that the local group of ρ is exactly D_{24}.

4.2 Braid group of $\mathcal{M}_{12}$ and its generating relations

In this section, we will introduce a representation of the orbifold fundamental group $\mathcal{M}_{12}$ which is the moduli space of the unordered distinct 12-tuples in $\mathbb{C}P^1$. Also, we will write down the generators and what path they represents explicitly, and those generators correspond to meridians in $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$. In fact, there is an orbifold isomorphism between $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ and $\mathcal{M}_{12}$.

For a Riemann surface Σ, we define $X(\Sigma)$ as Σ^n, the space of ordered n-tuples in Σ, where n is an integer which takes value 5 or 12 in this paper. The n-strand pure braid space of Σ consists of ordered n distinct points in Σ, denoted by $X^\circ(\Sigma)$.

The symmetric group S_n acts freely on $X^\circ(\Sigma)$, with quotient denoted by $Y^\circ(\Sigma)$, called the n-strand braid space of Σ. The fundamental group of $Y^\circ(\Sigma)$ is called the n-strand braid group of Σ, written $\mathrm{Br}_n(\Sigma)$. And the ordinary braid group Br_n refers to the case $\Sigma = \mathbb{C}$.

Setting $\zeta = e^{\frac{2\pi i}{n}}$, we will use the image of tuple $T = (1, \zeta, \zeta^2, \cdots, \zeta^{n-1})$ as the base point of $Y^\circ(\Sigma)$. Then a loop in $Y^\circ(\Sigma)$ base at T is a path in $X^\circ(\Sigma)$ from T to one of its S_n-images.

We now define specific braids $\rho_j \in \mathrm{Br}_n(\mathbb{C}^*)$, whose subscripts should be read modulo n, by specifying the motion of n points in the tuple T. The points ζ^{j-1} and ζ^j approach each other in a counterclockwise direction, and then continue to ζ^j and ζ^{j-1} respectively, while the remaining components do not move. For a precise definition, choose $r : [0,1] \to [1,\infty)$ continuous with $r(0) = r(1) = 1$ and $r(\frac{1}{2}) > 1$. Then ρ_j is the element in $\mathrm{Br}_n(\mathbb{C}^*)$ represented by the path $(x_0(t), \cdots, x_{n-1}(t))$ in $X^\circ(\mathbb{C}^*)$, where

$$\begin{aligned} x_{j-1}(t) &= \zeta^{j-1} r(t) e^{\frac{2\pi i t}{n}} \\ x_j(t) &= \zeta^j r(t)^{-1} e^{\frac{-2\pi i t}{n}} \\ x_k(t) &= \zeta^k \qquad\qquad \text{if } k \neq j-1, j. \end{aligned}$$

We define the "increasing" and "decreasing" words

$$I_j = \rho_j \rho_{j+1} \cdots \rho_{j+n-2} \qquad D_j = \rho_j \rho_{j-1} \cdots \rho_{j-n+2}$$

We have the following relation:

$$\begin{aligned} I_j \rho_k I_j^{-1} &= \rho_{k+1} \text{ for all } k \neq j-1, j-2 \\ D_j \rho_k D_j^{-1} &= \rho_{k-1} \text{ for all } k \neq j+1, j+2 \end{aligned}$$

which can be shown by picture-drawing.

The inclusions $\mathbb{C}^* \to (\mathbb{C}^* \cap \{0 \text{ or } \infty\}) \to \mathbb{C}P^1$ induce homomorphisms

$$\mathrm{Br}_n(\mathbb{C}^*) \to \mathrm{Br}_n(\mathbb{C}^* \cap \{0 \text{ or } \infty\}) \to \mathrm{Br}_n(\mathbb{C}P^1).$$

Sometimes we speak ρ_j as though they were elements of these other groups, meaning their images there.

Theorem 3 (Braid Groups)**.**

(1) *We have a fibration $Y^\circ(\mathbb{C}^*) \to \mathbb{C}^*$ by mapping the unordered tuple of numbers in $\mathbb{C}^*$ to their product. And the fiber is isomorphic to the group $\mathrm{Art}(\tilde{A}_{n-1})$, therefore $\mathrm{Br}_n(\mathbb{C}^*)$ is isomorpihc to the group $\mathrm{Art}(\tilde{A}_{n-1}) \rtimes \mathbb{Z}$.*

The part $\mathrm{Art}(\tilde{A}_{n-1})$ is generated by $\rho_0, \cdots, \rho_{n-1}$, with defining relations $\rho_j \rho_k \rho_j = \rho_k \rho_j \rho_k$ and $\rho_j \rho_k = \rho_k \rho_j$, according to whether $k \in \{j \pm 1\}$ or not.

(2) *Adjoining to the subgroup* $< \rho_0, \cdots, \rho_{n-1} >$ *the relations that all* D_j *coincide yields* $\mathrm{Br}_n(\mathbb{C}^* \cap \{0\})$. *Then* $D\rho_k D^{-1} = \rho_{k-1}$ *for all* k*, where* D *is the common image of all* D_j.

(3) *Adjoining to the subgroup* $< \rho_0, \cdots, \rho_{n-1} >$ *the relations that all* I_j *coincide yields* $\mathrm{Br}_n(\mathbb{C}^* \cap \{\infty\})$. *Then* $I\rho_k I^{-1} = \rho_{k+1}$ *for all* k*, where* I *is the common image of all* I_j.

(4) *Adjoining to* (1),(2) *and* (3) *the relation* $ID = 1$ *yields* $\mathrm{Br}_n(\mathbb{C}P^1)$.

Furthermore, we define the moduli space $\mathcal{M}_n$ of n-point subsets of $\mathbb{C}P^1$ as

$$\mathcal{M}_n = X^\circ(\mathbb{C}P^1)/(S_n \times \mathrm{PGL}_2\,\mathbb{C}) = Y^\circ(\mathbb{C}P^1)/\,\mathrm{PGL}_2\,\mathbb{C}$$

We assume $n \geq 3$ to avoid degenerate cases. Then $\mathrm{PGL}_2(\mathbb{C})$ acts freely on $X^\circ(C\mathbb{P}^1)$, so the quotient is a manifold. And thus $\mathcal{M}_n$ as the quotient of this manifold by S_{12}, it is an orbifold.

We take the image of the tuple T in $\mathcal{M}_n$ to be the basepoint for the orbifold fundamental group. Then the quotient map $Y^\circ(\mathbb{C}P^1) \to \mathcal{M}_n$ induces a surjective map $\mathrm{Br}_n(\mathbb{C}) \to \pi_1^{\mathrm{orb}}(\mathcal{M}_n)$ with kernel equal to the center of $\mathrm{Br}_n(\mathbb{C}P^1)$, which is isomorphic to $\mathbb{Z}/2$ with generator $I^n = D^{-n}$. Then we have the following theorem:

Theorem 4. *The orbifold fundamental group* $\pi_1^{orb}(\mathcal{M}_n, T)$ *is generated by* $\rho_0, \cdots, \rho_{n-1}$*, with defining relations:*

(1) $\rho_j\rho_k\rho_j = \rho_k\rho_j\rho_k$ *or* $\rho_j\rho_k = \rho_k\rho_j$*, according to whether* $k \in \{j \pm 1\}$ *or not.*

(2) *All the* $I_j := \rho_j\rho_{j+1}\cdots\rho_{j+n-1}$ *coincide, denoted by* I.

(3) *All the* $D_j := \rho_j\rho_{j-1}\cdots\rho_{j-n+1}$ *coincide, denoted by* D.

(4) $ID = 1$.

(5) $I^n = D^n = 1$.

4.3 The Deligne-Mostow 9-Ball

We have there is a Deligne-Mostow isomorphism between $(\mathbb{B}^9_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ and the moduli space $\mathcal{M}_{12}$ of unordered 12-tuples of distinct points in $\mathbb{C}P^1$.

However, this isomorphism is not very explicit. Allock and Basak manages to use the properties of the two orbifolds finding the relation between meridians and braids in [AB23], thus the generating relations of the meridians. This is a brilliant work.

We have defined ρ as the projection of $\tau \in \mathbb{B}^{13}$ to $\mathbb{B}^9_{\mathrm{DM}}$ in subsection 4.1, and we will use this as the basepoint for the orbifold fundamental group of $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}})$. And the mirrors of $s_0, \cdots, s_{11}$ are closest to ρ of all mirrors in $\mathcal{H}_{\mathrm{DM}}$ by lemma 2. Then we denote the meridians $M_{\rho,s_j} \in \pi_1^{\mathrm{orb}}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$ defined as in section 2.7 by ρ_j. In [AB23] Basak and Allcock state the main theorem:

Theorem 5. *There is a complex-analytic orbifold isomorphism between* $\mathcal{M}_{12}$ *and* $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{DM}$*, which identifies their respective basepoints* T *and* ρ*, and identifies the* $\rho_j \in \pi_1^{orb}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$ *with the elements of* $\pi_1^{orb}(\mathcal{M}_{12}, T)$ *denoted by the same symbols in section 4.2.*

And in particular, $\rho_0, \cdots, \rho_1 1$ *generate the orbifold fundamental group of* $\pi_1^{orb}((\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}, \rho)$*, with defining relations stated in the theorem 4.*

By this theorem, we perfectly understand the orbifold fundamental group of $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$, knowing the explicit generators and generating relations. In the next few sections, we will explain how this result is used to understand the orbifold fundamental group of the bigger one, $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$.

Basak and Allcock prove this theorem using the following fact:

Let X^{st} be the subset of X consisting of 12-tuples with no points of multiplicity > 5. And let $Y^{\mathrm{st}} = X^{\mathrm{st}}/S_{12}$

Here are the essential properties:

1. There is a complex algebraic variety isomorphism f from $Y^{\mathrm{st}}/\operatorname{PGL}_2(\mathbb{C})$ to $\mathbb{B}^9_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$. (Actually all we need is a homemorphism)

2. The restriction of f to $\mathcal{M}_{12} = Y^{\circ}/\operatorname{PGL}_2 \mathbb{C}$ is a complex analytic orbifold isomorphism, onto $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})$.

3. Those points of $Y^{\mathrm{st}}/\operatorname{PGL}_2(\mathbb{C})$ that are represented by 12-tuples with a single multiple point of multiplicity 2, are identified with the points of $\mathbb{B}^9_{\mathrm{DM}}$ that lie in exactly one component of $\mathcal{H}_{\mathrm{DM}}$.

4. $X^{\circ}/\operatorname{PGL}_2 \mathbb{C}$ resp. $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ is the covering space of $\mathcal{M}_{12} \cong (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}}$ corresponding to the subgroup of its orbifold fundamental

group that is generated by the squares resp. cubes of all meridians. (Meridians in $\pi_1^{\mathrm{orb}}((\mathbb{B}_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma_{\mathrm{DM}})$ correspond to the conjugacy class in π_1^{orb} that contains the standard braid generators.

We introduce the proof of theorem 5.

Lemma 3. *Every mirror of L that meets $\mathbb{B}^9_{\mathrm{DM}}$ is the mirror of a root of L_{DM}, except for the 40 mirrors which contain $\mathbb{B}^9_{\mathrm{DM}}$.*

Proof. Suppose $\mathbb{B}^9_{\mathrm{DM}}$ meets the mirror $r^\perp$ of a root r. Then the $\mathcal{E}$-span of r and L_4 is positive-definite (Recall $L = L_{\mathrm{DM}} \oplus L_4$). And all their inner products are divisible by θ, thus L_4 is a summand of the span. SInce r^2 and the minimal norm of L_4 are both 3, we must have $r \in L_4$ or $r \perp L_4$. □

Lemma 4. *The isomorphism f identifies the image of T in $\mathcal{M}_{12}$ with the image of ρ in $(\mathbb{B}^9_{\mathrm{DM}}) - \mathcal{H}_{\mathrm{DM}}/P\Gamma_{\mathrm{DM}}$.*

4.4 An Orbifold Fibration over the Deligne-Mostow Ball

We want to find a suitable neighborhood U of $\mathbb{B}^9_{\mathrm{DM}}$, invariant under the setwise stabilizer $P\Gamma^{\mathrm{SW}}_{\mathrm{DM}}$, thus we can have an orbifold fibration

$$(U - \mathcal{H})/P\Gamma^{\mathrm{SW}}_{\mathrm{DM}} \to (\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma$$

and derive the corresponding long exact sequence of orbifold homotopy groups which helps us to study $(U - \mathcal{H})/P\Gamma^{\mathrm{SW}}_{\mathrm{DM}}$ through $(\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}})/P\Gamma$.

Recall the roots $s_0, \cdots, s_{11}$ corresponds to the 12-gon is orthogonal to $s_A, \cdots, s_D$. This gives an orthogonal decomposition $L = L_4^{(1)} \oplus L_{\mathrm{DM}}$, thus a projection $\pi : \mathbb{B}^{13} \to \mathbb{B}^9_{\mathrm{DM}}$. Since $\mathcal{H}_{\mathrm{DM}}$ is the union of the mirrors of the roots of L_{DM}, and these are exactly the roots of L whose mirrors meet $\mathbb{B}^9_{\mathrm{DM}}$ except for those orthogonal to $\mathbb{B}^9_{\mathrm{DM}}$, i.e. lies in $L_4^{(1)}$, one can find the follow:

Lemma 5. *There is a $P\Gamma^{\mathrm{SW}}_{\mathrm{DM}}$-invariant neighborhood U of $\mathbb{B}^9_{\mathrm{DM}}$, such that orthogonal projection $\pi : \mathbb{B}^{13} \to \mathbb{B}^9_{\mathrm{DM}}$ realizes $U - \mathcal{H}$ as a fibration over $\mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ with fibers as follows. The fiber over each $x \in \mathbb{B}^9_{\mathrm{DM}} - \mathcal{H}_{\mathrm{DM}}$ is an open ball centered at x, in the $\mathbb{B}^4$ orthogonal to $\mathbb{B}^9_{\mathrm{DM}}$ at x, minus the 40 mirrors of L_4.*

Then one can take a basepoint $\sigma \in \overline{\tau\rho} - \{\rho\}$ close enough to ρ so that the meridians σ_j defined in 4.1 lie in U, for all $j = 0, \cdots, 11, A, \cdots, D$. Take this σ close enough to ρ, one is able make sense of the following orbifold fundamental group:

$$J := \pi_1^{\text{orb}}((U - \mathcal{H})/P\Gamma_{\text{DM}}^{\text{SW}}, \sigma)$$

The orthogonal projection π is $P\Gamma_{\text{DM}}^{\text{SW}}$-equivariant, inducing a map

$$(U - \mathcal{H})/P\Gamma_{\text{DM}}^{\text{SW}} \to (\mathbb{B}_{\text{DM}} - \mathcal{H}_{\text{DM}})/P\Gamma_{\text{DM}}$$

Since the later one is isomorphic to $\mathcal{M}_{12}$ as an orbifold, and the later one has contractible orbifold universal cover, a well-known result from [Bir75]. Then from the long exact homotopy sequence of orbifold fibration we derive a short exact sequence

$$1 \to \pi_1^{\text{orb}}(\text{fiber over } \rho, \sigma) \to J \to \pi_1^{\text{orb}}((\mathbb{B}_{\text{DM}}^9 - \mathcal{H}_{\text{DM}})/P\Gamma_{\text{DM}}, \rho) \to 1.$$

And the last term is just $\pi_1^{\text{orb}}(\mathcal{M}_{12}, T)$. About the first term, one can show that the fiber is just $(\mathbb{B} - \mathcal{H}/\text{Aut}(L_4), \sigma)$, where $\mathbb{B}$ and $\mathcal{H}$ are of the same construction of $\mathbb{B}^{13}$ and $\mathcal{H}$ where the lattice L is changed to L_4. Orlik and Soloman showed that its orbifold fundamental group is isomorphic to the standard 5-strand braid group $\text{Br}_5(\mathbb{C})$, and it's generated by the meridians $\sigma_A, \sigma_B, \sigma_C, \sigma_D$ subjected to the standard Artin relations of A_4. The latter fact is proved by computer check that $\sigma_A, \cdots, \sigma_D$ forms a standard set of generators for the braid group.

Thus we now have a exact sequence:

$$1 \to \text{Br}_5(\mathbb{C}) \to J \to \text{Br}_{12}(\mathbb{C}P^1)/(\mathbb{Z}/2) \to 1$$

where the first group is generated by $\sigma_A, \cdots, \sigma_D$ and the third group is generated by $\rho_0, \cdots, \rho_{11}$, and recall ρ is the projection of τ onto $\mathbb{B}_{\text{DM}}^9$, σ lies on $\overline{\tau\rho} - \rho$ close enough to ρ. Then the middle group J is a semidirect product of $\text{Br}_5(\mathbb{C})$ and $\text{Br}_{12}(\mathbb{C}P^1)/(\mathbb{Z}/2)$.

By theorem 4.4 of [Bas06], we know that meridians $\tau_0, \cdots, \tau_{11}, \tau_A, \cdots, \tau_D$ generate the orbifold fundamental group $G = \pi_1^{\text{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma, \tau)$ and they satisfy the Artin relations of the graph $A_4\widetilde{A_{11}}$. Then by theorem 2, we have $\sigma_0, \cdots, \sigma_{11}, \sigma_A, \cdots, \sigma_D$ generate G, and they satisfy the Artin relations. Also, these meridians generates J, and images of $\sigma_0, \cdots, \sigma_{11}$ in $\text{Br}_{12}(\mathbb{C}P^1)/(\mathbb{Z}/2)$

are precisely $\rho_0, \cdots, \rho_{11}$.

To determine J, we need two kinds of data of relations. First are the relations of $\mathrm{Br}_5(\mathbb{C})$, which we already know. The second part is that we need to find those relations in $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12})$ which are killed by taking quotient of J by the normal subgroup $\mathrm{Br}_5(\mathbb{C})$. That is to say that we should find relations of the form

$$\mathrm{word}(\sigma_0, \cdots, \sigma_{11}) = \mathrm{word}(\sigma_A, \cdots, \sigma_D) \tag{2}$$

where the left side is one of the relators defining $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12})$, except written with ρ_j in place of σ_j and the right side just lie in $\mathrm{Br}_5(\mathbb{C})$. The defining relations of $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12})$ all comes this way except those words which already vanished in J, though they are still of the form $\mathrm{word}(\cdots) = 1$.

Thus the defining relations for J consists of the Artin relators of $\sigma_A, \cdots, \sigma_D$ and plus relations of the form of equation 2. The later relations correspond to defining relators of $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12})$. Those are enough to define J. To find the later relations, we just need to write relators in $\pi_1^{\mathrm{orb}}(\mathcal{M}_{12})$ as a product of $\sigma_A, \cdots, \sigma_D$ inside J. In this way, Allcock and Basak prove the following theorem, stating all relations needed to define J:

Theorem 6 (Allcock-Basak, 2023)**.** *J has generators $\sigma_A, \cdots, \sigma_D, \sigma_0, \cdots, \sigma_{11}$ subjected to the following defining relations:*

(1) *the Artin relations of the $\widetilde{A_{11}}A_4$ diagram;*

(2) *All the $I_j := \sigma_j \cdots \sigma_{j+11}$ coincide, denoted by I;*

(3) *All the $D_j := \sigma_j \sigma_{j-1} \cdots \sigma_{j-11}$ coincide, denoted by D;*

(4) *$ID = \Delta(\sigma_A, \cdots, \sigma_D)^2$;*

(5) *$I^6 = D^6$.*

Furthermore, for all $k = 0, \cdots, 11$ one has $I_k \sigma_k I^{-1} = \sigma_{k+1}$, $D\sigma_k D^{-1} = \sigma_{k-1}$ and $\Delta(\sigma_1, \cdots, \sigma_{11}) \sigma_k \Delta(\sigma_1, \cdots, \sigma_{11})^{-1} = \sigma_{12-k}$.

where in the theorem Δ is defined as

$$\Delta(g_1, \cdots, g_m) = (g_1 \cdots g_m)(g_1 g_2 \cdots g_{m-1}) \cdots (g_1 g_2) g_1.$$

4.5 Proof of the Main Theorem

In this section, the writer is going to show the last part of proof of Allcock and Basak. We have know the explicit form of $J = \pi_1^{\mathrm{orb}}((U - \mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{SW}})$, but there is still some distance to understand $G = \pi_1^{\mathrm{orb}}((\mathbb{B}^{13} - \mathcal{H})/P\Gamma)$.

However, Allcock and Basak notice that the local group at σ inside the two orbifolds $(U - \mathcal{H})/P\Gamma_{\mathrm{DM}}^{\mathrm{SW}}$ and $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ are the same, since the full $P\Gamma$-stabilizer of σ preserves $\mathbb{B}_{\mathrm{DM}}^{9}$, hence U. Allcock and Basak have the following result on this group:

Lemma 6. *The local group at σ is a dihedral of order 24, generated by*

$$I \cdot \Delta(\sigma_1, \cdots, \sigma_{11})^{-1}$$

of order 12, and

$$I^6 \cdot \Delta(\sigma_1, \cdots, \sigma_{11})^{-1}$$

of order 2. *The second inverts the first.*

Notice that theorem 6 holds for any choice of the $A_4\widetilde{A_{11}}$ subdiagram of the graph $P_2\mathbb{F}_3$, then by varying our choice, we get the following theorem:

Theorem 7 (Allcock-Basak, 2023)**.** *The group G is the quotient of* $\mathrm{Art}(P^2\mathbb{F}_3) \rtimes \mathrm{Aut}(P^2\mathbb{F}_3)$ *by the following relations and possibly some unknown relations. (Here base point is changed to τ).*

For any $\widetilde{A_{11}}A_4$ subdiagram of $P^2\mathbb{F}_3$ and any labeling of the nodes of $\widetilde{A_{11}}$ by $\tau_0, \cdots, \tau_{11}$, cyclically around it in either direction, and either labeling of the nodes of A_4 by $\tau_A, \cdots, \tau_D$ along it. Then

(1) $\tau_1\tau_2\cdots\tau_{11} \cdot \Delta(\tau_A, \cdots, \tau_D)^{-1}$ *equals the element of* $\mathrm{Aut}(P^2\mathbb{F}_3)$ *that permutes the nodes of the $\widetilde{A_{11}}$ by $\tau_j \to \tau_{j+1}$, and those of the A_4 by $\tau_A \leftrightarrow \tau_D$ and $\tau_B \leftrightarrow \tau_C$.*

(2) $(\tau_1\tau_2\cdots\tau_{11})^6 \cdot \Delta(\tau_1, \cdots, \tau_{11})^{-1}$ *equals the element of* $\mathrm{Aut}(P^2\mathbb{F}_3)$ *that permutes the nodes of the $\widetilde{A_{11}}$ by $\tau_j \leftrightarrow \tau_{6-j}$, and fixes each of $\tau_A, \cdots, \tau_D$.*

Proof. Due to theorem 4.4 of [Bas06], one has G is generated by all meridians, and they satisfy the Artin relations.

The segment $\overline{\sigma\tau}$ identifies the orbifold fundamental groups of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ based at ρ and τ. The segment is fixed point wise by the $P\Gamma$-stabilizer

D_{24} of σ. And thus the isomorphism identifies the local group at σ with a subgroup of $L_3(3) \rtimes 2$. And also, it identifies $\sigma_0, \cdots, \sigma_{11}, \sigma_A, \cdots, \sigma_D$ with $\tau_0, \cdots, \tau_{11}, \tau_A, \cdots, \tau_D$. Thus the words in lemma 6 are identified with a subgroup D_{24} inside $L_3(3) \rtimes 2$. Then the conjugate actions of these words corresponds to some conjugation actions inside $L_3(3) \rtimes 2$, so these conjugate actions permutes the 26 line- and point-meridians. Since the pointwise $(L_3(3) \rtimes 2)$-stabilizer of an $\widetilde{A_{11}}A_4$ diagram is trivial. Therefore the conjugate action of these words' actions on the 26 generators are completely determined by their actions on $\tau_0, \cdots, \tau_{11}, \tau_A, \cdots, \tau_D$ (This can be understood from the fact that the corresponding roots generate the lattice L).

To check (1) and (2), through the change of basepoint isomorphism on the orbifold fundamental groups of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$, Then we just need to work out the conjugate actions of words in lemma 6 on elements $\sigma_0, \cdots, \sigma_{11}, \sigma_A, \cdots, \sigma_D$. This is not hard to check. One can just uses the relations in theorem 6 to varify, noticing $\sigma_0, \cdots, \sigma_{11}$ and $\sigma_A, \cdots, \sigma_D$ commutes the same time.

This established the lemma for one particular choice of roots $s_0, \cdots, s_{11}$, and $s_A, \cdots, s_D$ in section 4.1. This does not quite prove the theorem, because $L_3(3) \rtimes 2$ acts with two orbits on the set of such choices. The other orbit can be represented by the labeling $s_0, \cdots, s_{11}$ and $s_D, \cdots, s_A$ corresponding to meridians $\tau_0, \cdots, \tau_{11}, \tau_D, \cdots, \tau_A$. This case could be proved from the first, by the relation $\Delta(\tau_A, \cdots, \tau_D) = \Delta(\tau_D, \cdots, \tau_A)$ in $\mathrm{Br}_5(\mathbb{C})$. □

Remark 2. *In this theorem, we see G as a quotient of $\mathrm{Art}(P^2\mathbb{F}_3) \rtimes \mathrm{Aut}(P^2\mathbb{F}_3)$, i.e. the part where the meridians satisfies the Artin relations, and take semidirect product with the local group at τ. Because this is more convienient for us to describe words in (1),(2), we did not view this directly as a quotient of $\mathrm{Art}(P^2\mathbb{F}_3)$.*

Before the final step of the proof, we introduce the proof that G is larger than or at least $\mathbb{Z}_2$. It is equivalent to show that $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ has a connected orbifold double cover.

Lemma 7. *$(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$ has a connected orbifold double cover.*

Proof. By theorem 7.1 in [All00], there is a degree 4 holomorphic automorphic form for $P\Gamma$, whose zero locus is $\mathcal{H}$ with multiplicity 1 along each component. This means that there exists a holomorphic function Ψ_0 on the preimage Ω of $\mathbb{B}^{13}$ in $\mathbb{C}^{14} - \{0\}$, which is of degree -4, and has a simple zero along each component of the preimage $\tilde{\mathcal{H}}$ of $\mathcal{H}$ in Ω, and no other zeros. Also, the same reference shows that ψ_0 transforms by a character $\Gamma \to \mathbb{Z}/3$.

Then let Z be the space of pairs (l, ϕ) where l is the preimage in $\mathbb{C}^{14} - \{0\}$ of a point of $\mathbb{B}^{13} - \mathcal{H}$, and φ is a holomorphic function which its value on l has square $\Psi_0^3|_l$. This is a degree 2 cover of $\mathbb{B}_{13} - \mathcal{H}$, and it is connected beacuse Ψ_0^3 has zeros of odd order along $\tilde{\mathcal{H}}$. Since Ψ_0^3 is Γ-invariant, this yields a Γ action on Z: given $g \in \Gamma$, there is a map sending (l, ϕ) to $(g(l), \phi \circ g^{-1})$. Moreover, φ is homogeneous of degree -6, thus the scalars in Γ act trivially. Then we get a $P\Gamma$-action on Z and a $P\Gamma$-equivariant map $Z \to \mathbb{B}^{13} - \mathcal{H}$. Then $Z/P\Gamma$ is an orbifold double cover of $(\mathbb{B}^{13} - \mathcal{H})/P\Gamma$. □

From this lemma, we can see that G has a $\mathbb{Z}_2$ quotient, thus G/S still has a $\mathbb{Z}_2$ quotient where S is the normal subgroup generated by squares of all meridians.

We now come to the proof of theorem 1.

Proof. Consider $\delta = \tau_0\tau_1 \cdots \tau_{10}(\tau_1\tau_2 \cdots \tau_{11})$, as an element inside $\mathrm{Art}(\widetilde{A_{11}}) \subset \mathrm{Art}(P^2\mathbb{F}_3)$. Then by theorem 7 (1), we have δ dies in G (just replace $\tau_0, \cdots, \tau_{11}$ by $\tau_1, \cdots, \tau_{11}, \tau_0$). We write $\bar{\delta}$ for the image of δ in $\mathrm{Cox}(\widetilde{A_{11}}) \subset \mathrm{Cox}(P^2\mathbb{F}_3)$.

One could prove that the subgroup of $\mathrm{Cox}(\widetilde{A_{11}}) \cong \mathbb{Z}^{11} \rtimes S_{12}$ normally generated by $\bar{\delta}$ is the translation subgroup $\mathbb{Z}_{11}$. Then G/S is a quotient of $\mathrm{Cox}(P^2\mathbb{F}_3)$ where all $\bar{\delta}$ for all choices of $\widetilde{A_{11}} \subset P_2\mathbb{F}_3$ dies. We could say that G/S is a quotient of the deflation of $\mathrm{Cox}(P_2\mathbb{F}_3)$. Here we quote a result of Conway and Simons [CS01], that the "deflation" of $\mathrm{Cox}(P^2\mathbb{F}_3)$, i.e. the quotient of this group by the subgroup normally generated by the translation subgroups of the $\mathrm{Cox}(\widetilde{A_{11}})$'s coming from all the $\widetilde{A_{11}}$ subdiagrams of $P_2\mathbb{F}_3$. They proved that this quotient is the bimonster $(M \times M) \rtimes 2$. Thus G/S is a quotient of $(M \times M) \rtimes 2$. Since M is simple, and by lemma 7, G/S is either the bimonster or just $\mathbb{Z}/2$. □

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