Seminar on Bridgeland Stability Conditions

Existence of stability conditions on projective variety

April 1, 2026

1 Preliminaries

1.1 Basic concepts of stability conditions

1.2 Bayer property and restriction-N property

The following relation between stability conditions, introduced in [Li25], will be useful in our argument.

Definition 1.1. For two pre-stability conditions σ, ω on $\mathcal{T}$, we define

$$\sigma \precsim \omega \iff \mathcal{P}_\sigma(\theta) \subset \mathcal{P}_\omega(< \theta) \quad \text{for every } \theta \in \mathbb{R}$$
$$\sigma \precapprox \omega \iff \mathcal{P}_\sigma(\theta) \subset \mathcal{P}_\omega(\leq \theta) \quad \text{for every } \theta \in \mathbb{R}$$

As a remark, the relation is transitive.

Lemma 1.2 (Lemma 2.2, [Li25])**.** *Let $\sigma, \omega \in Stab(\mathcal{T})$ and Φ be an exact autoequivalence on $\mathcal{T}$. Then the followings are equivalent:*

(1) $\sigma \precapprox \omega$.

(2) For every σ-stable object $E \in \mathcal{T}$, $\phi^+_\omega(E) \leq \phi_\sigma(E)$.

(3) For every ω-stable object $E \in \mathcal{T}$, $\phi_\omega(E) \leq \phi^-_\sigma(E)$.

(4) $\Phi(\sigma) \precapprox \Phi(\omega)$.

All statements hold for $\precsim$ by replacing $\leq$ with $<$.

Lemma 1.3. *Let σ be a stability condition on $C \times M$, where C is a smooth projective curve with genus $g \geq 1$. For any closed point $q \in C$ and $H = \{q\} \times M$, we have*

$$\sigma \precapprox \sigma \otimes \mathcal{O}(H).$$

Proof. Let $\iota : H \to C \times M$ denote the closed inclusion. For any σ-stable object F, we start with the standard short exact sequence of sheaves

$$0 \to \mathcal{O}_{C\times M}(-H) \to \mathcal{O}_{C\times M} \to \iota_*\mathcal{O}_H \to 0$$

Tensoring with F, we obtain the following distinguished triangle in $D^b(C \times M)$:

$$\iota_*\iota^*F[-1] \longrightarrow F \otimes \mathcal{O}(-H) \longrightarrow F \longrightarrow \iota_*\iota^*F$$

Claim: $\phi^+_\sigma(\iota_*\iota^*F) \leq \phi_\sigma(F) + 1$.

Proof of the Claim. Let G be the HN-factor of $\iota_*\iota^*F$ with respect to σ, so by definition,

$$\phi^+_\sigma(\iota_*\iota^*F) = \phi_\sigma(G).$$

By [LMP+25, Remark 2.24], the group $Pic^0(C)$ acts trivially on the stability condition σ. As the object $\iota_*\iota^*F$ is supported on $H = \{q\} \times M$, the group $Pic^0(C)$ acts trivially on $\iota_*\iota^*F$. Because the HN filtration is canonical and unique, every HN factor of $\iota_*\iota^*F$ must also be fixed by the $Pic^0(C)$ action. In particular, the first HN factor G is supported on H as well.

Applying the functor $\mathrm{Hom}(G, -)$, we obtain a long exact sequence of vector spaces:

$$\cdots \longrightarrow \mathrm{Hom}(G, F) \longrightarrow \mathrm{Hom}(G, \iota_*\iota^*F) \longrightarrow \mathrm{Hom}(G, F \otimes \mathcal{O}(-H)[1]) \longrightarrow \dots$$

Because G is supported on H, we have $G \cong G \otimes \mathcal{O}_H$, and since H is pullback of a point on a curve, its normal bundle is trivial, which implies $\mathcal{O}(-H)|_H \cong \mathcal{O}_H$, so $G \otimes \mathcal{O}(H) \cong G$, which makes us easily identify $\mathrm{Hom}(G, F \otimes \mathcal{O}(-H)[1]) \cong \mathrm{Hom}(G, F[1])$.

By the property of HN filtrations, $\mathrm{Hom}(G, \iota_*\iota^*F) \neq 0$, so either

$$\mathrm{Hom}(G, F) \neq 0 \quad \text{or} \quad \mathrm{Hom}(G, F[1]) \neq 0$$

Since both G and F are σ-semistable objects, the existence of a non-zero morphism implies a bound on their phases:

$$\text{Either } \phi_\sigma(G) \leq \phi_\sigma(F) \quad \text{or} \quad \phi_\sigma(G) \leq \phi_\sigma(F[1]) = \phi_\sigma(F) + 1.$$

In either case, the weaker inequality $\phi_\sigma(G) \leq \phi_\sigma(F) + 1$ always holds. This proves the claim. □

Returning to the proof of the lemma, we need to estimate the maximum phase of $F \otimes \mathcal{O}(-H)$ with respect to σ, by the distinguished triangle above,

$$\phi^+_{\sigma\otimes\mathcal{O}(H)}(F) \leq \max\{\phi_\sigma(F), \phi^+_\sigma(\iota_*\iota^*F) - 1\} = \phi_\sigma(F)$$

Since this inequality holds for *every* σ-stable object F, it immediately follows that

$$\sigma \preceq \sigma \otimes \mathcal{O}(H),$$

which concludes the proof. □

Now we come to restrict N-property.

Let Y be a smooth projective variety and $X \in |D|$ be a smooth subvariety of Y for some divisor D on Y. Denote by $\iota : X \hookrightarrow Y$ the inclusion morphism, $\iota_* : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ the pushforward functor, ι^* the derived pull-back functor. The induced map $[\iota_*] : \mathrm{K_{num}}(X) \to \mathrm{K_{num}}(Y) : [E] \mapsto [\iota_*E]$ is well-defined.

Definition 1.4. [Li25, Definition 6.3] Let $\mathcal{A}$ be the heart of a bounded t-structure on $D^b(Y)$, we denote by

$$\mathcal{A}|_{D^b(X)} := \{E \in D^b(X) : \iota_*E \in \mathcal{A}\}$$

the full subcategory in $D^b(X)$.

Proposition 1.5. *[Li25, Proposition 6.4] Let X, Y as above. Let $\sigma = (\mathcal{A}, Z)$ be a stability condition on $D^b(Y)$ satisfying*

$$\sigma \otimes \mathcal{O}_Y(D) \precsim \sigma[1].$$

Then

$$\sigma|_{D^b(X)} := (\mathcal{A}|_{D^b(X)}, Z \circ [\iota_*])$$

is a stability condition on $D^b(X)$.

Moreover, an object $E \in D^b(X)$ *is* $\sigma|_{D^b(X)}$*-(semi)stable if and only if* $\iota_* E$ *is* σ*-(semi)stable.*

If $\sigma \otimes \mathcal{O}_Y(D') \lesssim \sigma[1]$ *(resp.* $\sigma \lessapprox \sigma \otimes \mathcal{O}_Y(D')$*) for some divisor* D'*, then the restricted stability condition also satisfies* $\sigma|_{D^b(X)} \otimes \mathcal{O}_X(D') \lesssim \sigma|_{D^b(X)}[1]$ *(resp.* $\sigma|_{D^b(X)} \lessapprox \sigma|_{D^b(X)} \otimes \mathcal{O}_X(D')$*).*

Remark 1.6. $\sigma[1]$ is the shift of σ by 1 defined as follow:

1. $\mathcal{P}_{[1]}(\phi) := \mathcal{P}(\phi + 1)$ for all $\phi \in \mathbb{R}$. Consequently, $\mathcal{A}_{\sigma[1} := \mathcal{P}_{[1]}((0, 1]) = \mathcal{P}((1, 2])$.
2. $Z_{[1]}(E) := e^{-i\pi} Z(E) = -Z(E)$ for any object E.

Remark 1.7. It is worth noting that a semistable object $E \in \mathcal{P}(\phi)$ with respect to σ remains semistable with respect to $\sigma[1]$, but its new phase becomes $\phi - 1$. This structural consistency is fundamental to tracking Harder-Narasimhan filtrations across the stability manifold.

Remark 1.8. Please note that the proof of Proposition 6.4 in the current arXiv versions contains an error, which assumes for any $m < \dim X = n$, there exist a subvariety Y of $\dim m$. The corrected and fully accurate version of this proof can be found in the updated manuscript available on the author's personal homepage.

1.3 Functorial operations along finite morphisms

Let $f : Y \to X$ be a finite morphism between smooth projective varieties.

Definition 1.9. Given a pre-stability condition $\sigma_Y = (\mathcal{P}, Z)$ on $D^b(Y)$, define its pushforward to $D^b(X)$ as

$$f_\# \sigma_Y := (f_\# \mathcal{P}, f_\# Z),$$

where

$$f_\# \mathcal{P}(\theta) := \{E \in D^b(X) \mid f^* E \in \mathcal{P}(\theta)\} \quad \text{for every } \theta \in \mathbb{R},$$
$$f_\# Z := Z \circ f^* : K_0(X) \xrightarrow{f^*} K_0(Y) \to \mathbb{C}.$$

Conversely, given a pre-stability condition $\sigma_X = (\mathcal{P}, Z)$ on $D^b(X)$, we define its pullback to $D^b(Y)$ as

$$f^! \sigma_X := (f^! \mathcal{P}, f^! Z),$$

where

$$f^! \mathcal{P}(\theta) := \{E \in D^b(Y) \mid f_* E \in \mathcal{P}(\theta)\}, \quad \text{for every } \theta \in \mathbb{R},$$
$$f^! Z := Z \circ f_* : K_0(Y) \xrightarrow{f_*} K_0(X) \to \mathbb{C}.$$

Lemma 1.10. *Assume that* $f_\# \sigma$ *is a pre-stability condition on* $D^b(X)$*. Then for any line bundle* $\mathcal{L}$ *on* X*, we have*

$$(f_\# \sigma) \otimes \mathcal{L} = f_\# (\sigma \otimes f^* \mathcal{L}).$$

Proof. we prove by verifying their slicings and central charges coincide. Direct computation proves

$$\mathcal{P}_{(f_\# \sigma) \otimes \mathcal{L}}(\theta) = \{F \otimes \mathcal{L} \mid f^* F \in \mathcal{P}_\sigma(\theta)\}$$

similarly check proves it is equivalent to $F \in \mathcal{P}_{f_\# (\sigma \otimes f^* \mathcal{L})}(\theta)$.

Using definition one can also check

$$Z_{(f_\# \sigma)\otimes \mathcal{L}}(E) = Z_\sigma(f^*E \otimes f^*\mathcal{L}^{-1}) = Z_{f_\#(\sigma \otimes f^*\mathcal{L})}(E)$$

This completes the proof. □

Lemma 1.11. *Let $\sigma \leq \omega$ be pre-stability conditions. Assume that both $f_\#\sigma$ and $f_\#\omega$ (respectively, $f^!\sigma$ and $f^!\omega$) are pre-stability conditions. Then*

$$f_\#\sigma \leq f_\#\omega \quad (\textit{respectively, } f^\#\sigma \leq f^\#\omega).$$

Proof. We will prove the statement for the pushforward $f_\#$, same argument shows $f^\#$.

To prove $f_\#\sigma \leq f_\#\omega$, we will show that $\mathcal{P}_{f_\#\sigma}(\theta) \subset \mathcal{P}_{f_\#\omega}(\leq \theta)$.

By the definition of $f_\#\sigma$, the condition $F \in \mathcal{P}_{f_\#\sigma}(\theta)$ means that f^*F is σ-semistable of phase θ.

Now, consider the HN-filtration of F with respect to the stability condition $f_\#\omega$ on $D^b(X)$:

$$0 = F_0 \to F_1 \to \cdots \to F_s = F$$

And $G_j = Cone(F_{j-1} \to F_j)$ denotes the HN factors, so $G_j \in \mathcal{P}_{f_\#\omega}(\theta_j)$ with strictly decreasing phases:

$$\theta_1 > \theta_2 > \cdots > \theta_s$$

which means $f^*G_j \in \mathcal{P}_\omega(\theta_j)$ for all j.

Since f^* is an exact functor on triangulated categories, this gives a filtration of f^*F:

$$0 = f^*F_0 \to f^*F_1 \to \cdots \to f^*F_s = f^*F$$

and it is *precisely* the Harder-Narasimhan filtration of f^*F with respect to ω. In particular, $\phi^+_\omega(f^*F) = \phi_\omega(f^*G_1) = \theta_1$, and $\sigma \leq \omega$ implies

$$\theta_1 = \phi^+_\omega(f^*F) \leq \phi_\sigma(f^*F) = \theta$$

which means $F \in \mathcal{P}_{f_\#\omega}(\leq \theta)$.

That finally completes the proof. □

Proposition 1.12 (Proposition 3.4, [Li26])**.** *Let $f : Y \to X$ be a surjective finite morphism between smooth projective varieties, and let σ be a stability condition on $D^b(Y)$*[1]*. Assume that*

$$f^*f_*\mathcal{P}_\sigma(\theta) \subset \mathcal{P}_\sigma(\leq \theta), \quad \forall \theta \in \mathbb{R}$$

Then $f_\#\sigma$ is a stability condition on $D^b(X)$.

This result comes from a non-trivial application of [Pol07, Theorem2.1.2] which state as:

Let $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$ be a pair of triangulated categories in which all small coproducts exist. Let $\mathcal{D}_1 \subset \tilde{\mathcal{D}}_1$ and $\mathcal{D}_2 \subset \tilde{\mathcal{D}}_2$ be full triangulated essentially small subcategories. Assume we have an exact functor $\Phi : \tilde{\mathcal{D}}_1 \to \tilde{\mathcal{D}}_2$ commuting with small coproducts that admits a left adjoint functor $\Psi : \tilde{\mathcal{D}}_2 \to \tilde{\mathcal{D}}_1$. Assume further that $\Phi(\mathcal{D}_1) \subset \mathcal{D}_2$ and $\Psi(\mathcal{D}_2) \subset \mathcal{D}_1$.

[1] Note: The original text in the preprint states "σ on $D^b(X)$", which is a typographical error. The pushforward $f_\#\sigma$ requires σ to be defined on the domain Y. The proof in the paper also uses $\mathcal{D}_1 = D^b(X)$ and $\mathcal{D}_2 = D^b(Y)$, taking σ on Y.

Theorem 1.13. • *Let $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$ be a t-structure on $\mathcal{D}_2$ such that the functor $\Phi\Psi : \mathcal{D}_2 \to \mathcal{D}_2$ is right t-exact. Assume in addition that $\mathcal{D}_1 = \Phi^{-1}(\mathcal{D}_2)$, i.e., for any object $F \in \mathcal{D}_1$ such that $\Phi(F) \in \mathcal{D}_2$, one has $F \in \mathcal{D}_1$. Then there exists a (unique) t-structure on $\mathcal{D}_1$ with*

$$\mathcal{D}_1^{\geq 0} = \{F \in \mathcal{D}_1 | \Phi(F) \in \mathcal{D}_2^{\geq 0}\}$$

Moreover, the functor Φ is t-exact with respect to these t-structures.

- *Assume also that for any $F \in \mathcal{D}_1$ such that $\Phi(F) = 0$, one has $F = 0$. Then*

$$\mathcal{D}_1^{\leq 0} = \{F \in \mathcal{D}_1 | \Phi(F) \in \mathcal{D}_2^{\leq 0}\}$$

. In this situation, if $\mathcal{C}_2$ is Noetherian, then so is $\mathcal{C}_1$, where $\mathcal{C}_i = \mathcal{D}_i^{\leq 0} \cap \mathcal{D}_i^{\geq 0}$.

2 Stability Conditions on Products of Elliptic Curves

2.1 Baseline results on Abelian varieties

Let $E^n := E_1 \times \cdots \times E_n$ be the product of n smooth elliptic curves, with $E_i \cong E$. For each $1 \leq i \leq n$, fix a point $q_i \in E_i$ and define

$$H_i := E_1 \times \cdots \times E_{i-1} \times \{q_i\} \times E_{i+1} \times \cdots \times E_n \subset E^n.$$

By abuse of notation, we also denote by H_i the numerical first Chern class of $\mathcal{O}_{E^n}(H_i)$. Set

$$H := H_1 + \cdots + H_n.$$

We define a lattice $(\Lambda \cong \mathbb{Z}^{2^n}, v)$ for $K_0(E^n)$ by

$$v : K_0(E^n) \to \Lambda, \quad [F] \mapsto (H_{j_1} \dots H_{j_s} ch_{n-s}(F))_{1 \leq j_1 < \cdots < j_s \leq n}.$$

Theorem 2.1. *Let σ be a stability condition on E^n with respect to the lattice (Λ, v). Then:*

1. *[FLZ22, Theorem 1.1, Proposition 2.9] all skyscraper sheaves $\{\mathcal{O}_p\}_{p \in E^n}$ are σ-stable with the same phase.*
2. *[LMP$^+$25, Theorem 1.1] the map*

$$Stab_{(\Lambda,v)}(E^n) \longrightarrow Hom_{\mathbb{Z}}(\Lambda, \mathbb{C}) \times \mathbb{R}, \quad \sigma = (\mathcal{P}, Z) \mapsto (Z, \phi_\sigma(\mathcal{O}_p))$$

is injective.

For every $(a, b) \in \mathbb{Q}_{>0} \times \mathbb{Q}$, define a group homomorphism $Z^{a,b} : \Lambda \to \mathbb{C}$ by

$$\begin{aligned} Z^{a,b}(v(F)) &= -\int_{E^n} e^{-(b+ia)H} ch(F) \\ &= -ch_n(F) + (b + ia)Hch_{n-1}(F) - \cdots + (-1)^{n+1}\frac{(b + ia)^n}{n!} H^n rk(F). \end{aligned}$$

Theorem 2.2. *[Liu21, Theorem 5.9], [LMP$^+$25, Theorem 4.5] For any $(a, b) \in \mathbb{Q}_{>0} \times \mathbb{Q}$, there exists a stability condition*

$$\sigma^{a,b} = (\mathcal{P}^{a,b}, Z^{a,b}) \in Stab_{(\Lambda,v)}(E^n)$$

*with central charge $Z^{a,b}$ as in (**??**) and $\phi_{\sigma^{a,b}}(\mathcal{O}_p) = 1$. Such a stability condition $\sigma^{a,b}$ is unique by 2.1.*

For a positive integer m, denote the self-isogeny

$$\pi_m : E^n \longrightarrow E^n, \quad z \mapsto mz. \tag{1}$$

Lemma 2.3. *[Li26, Lemma 5.3] Let σ be a stability condition on E^n. Then both $\pi_m^! \sigma$ and $(\pi_m)_\# \sigma$ are stability conditions on E^n.*

Proof. By [LMP+25, Remark 2.24], the group $E^n \times Pic^0(E^n)$ acts trivially on $Stab_{(\Lambda,v)}(E^n)$. In particular, for any $F \in \mathcal{P}_\sigma(\theta)$, $\mathcal{L} \in Pic^0(E^n)$ and $g \in E^n$, we have

$$g^*(F \otimes \mathcal{L}) \in \mathcal{P}_\sigma(\theta).$$

Hence, for any $F \in \mathcal{P}_\sigma(\theta)$, we have

$$(\pi_m)_* \pi_m^* F = \bigoplus_{\mathcal{L}^m \cong \mathcal{O}} F \otimes \mathcal{L} \in \mathcal{P}_\sigma(\theta).$$

Similarly,

$$\pi_m^* (\pi_m)_* F = \bigoplus_{g \in Torsion(E^n)} g^* F \in \mathcal{P}_\sigma(\theta).$$

□

Remark 2.4. The decomposition of $(\pi_m)_* \pi_m^* F$ and $\pi_m^* (\pi_m)_* F$ comes from the following:

The first result comes from

$$(\pi_m)_* \pi_m^* F \cong F \otimes (\pi_m)_* \mathcal{O}_{E^n}$$

by projection formula and

$$(\pi_m)_* \mathcal{O}_{E^n} \cong \bigoplus_{\mathcal{L}^m \cong \mathcal{O}} \mathcal{L}$$

the right hand side is the direct sum of all elements in $\mathrm{Ker}(\widehat{\pi_m} : \mathrm{Pic}^0(E^n) \to \mathrm{Pic}^0(E^n))$ where $\widehat{\pi_m}$ is the dual isogeny.

The second identity is proved by 2 steps, first using flat base change theorem with respect to

$$\begin{array}{ccc} E^n \times_{E^n} E^n & \xrightarrow{p_2} & E^n \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_m} \\ E^n & \xrightarrow[\pi_m]{} & E^n \end{array}$$

we get $\pi_m^* (\pi_m)_* F \cong (p_1)_* p_2^* F$ using the isomorphism

$$E^n \times G \xrightarrow{\sim} E^n \times_{E^n} E^n$$

$$(x, g) \mapsto (x, x + g)$$

where $G = \ker(\pi_m) = E^n[m]$. Since $E^n \times_{E^n} E^n$ splits into disjoint connected components indexed by $g \in \ker(\pi_m)$, the functor $(p_1)_* p_2^*$ splits into a direct sum over these components, so

$$\pi_m^* (\pi_m)_* F \cong \bigoplus_{g \in \ker(\pi_m)} g^* F$$

Lemma 2.5. *Let $\pi_m : E^n \to E^n$ and $\sigma^{a,b} = (\mathcal{P}^{a,b}, Z^{a,b})$ as above, and $\mathcal{L}$ be a line bundle on E^n whose numerical first Chern class is $ch_1(\mathcal{L}) = H$. Then the following relations hold:*

1. *For the central charges: $\pi_{m\#} Z^{a,b} = m^{2n} Z^{a/m^2, b/m^2}$ and $\pi_m^{\#} Z^{a,b} = Z^{m^2 a, m^2 b}$.*

2. *For the stability conditions:* $\pi_{m\#}\sigma^{a,b} = m^{2n}\sigma^{a/m^2,b/m^2}$ *and* $\pi_m^{\#}\sigma^{a,b} = \sigma^{m^2a,m^2b}$.
3. *For the tensor action by* $\mathcal{L}$*:* $\sigma^{a,b}\otimes\mathcal{L} = \sigma^{a,b+1}$.

Proof.

- By definition, the central charge is given by the integral formula:

$$Z^{a,b}(v(F)) = -\int_{E^n} e^{-(b+ia)H}ch(F) = \sum_{j=0}^{n}(-1)^{n+1-j}\frac{(b+ia)^{n-j}}{(n-j)!}H^{n-j}ch_j(F)$$

so using $\pi_{m\#}Z^{a,b}(F) = Z^{a,b}(\pi_m^*F)$ and $H^{n-j}ch_j(\pi_m^*F) = m^{2j}H^{n-j}ch_j(F)$ so direct expansion shows

$$Z^{a,b}(\pi_{m*}F) = m^{2n}Z^{a/m^2,b/m^2}(v(F))$$

and similarly since $\pi_m^{\#}Z^{a,b}(F) = Z^{a,b}(\pi_{m*}F)$ and $H^{n-j}ch_j(\pi_{m*}F) = m^{2n-2j}H^{n-j}ch_j(F)$, the same expansion gives

$$Z^{a,b}(\pi_m^*F) = m^{2n}Z^{a/m^2,b/m^2}(v(F))$$

- By Theorem 2.1, since π_{m*} and π_m^* does not change the phase of skyscraper sheaves, then Z uniquely determines the stability condition, that is $\sigma^{a',b'} = \sigma^{a,b}$ if and only if $Z^{a',b'} = Z^{a,b}$ Therefore, $\pi_{m\#}\sigma^{a,b} = m^{2n}\sigma^{a/m^2,b/m^2}$ and $\pi_m^{\#}\sigma^{a,b} = \sigma^{m^2a,m^2b}$. The corresponding slicings satisfy $\pi_{m\#}\mathcal{P}^{a,b} = \mathcal{P}^{a/m^2,b/m^2}$ and $\pi_m^{\#}\mathcal{P}^{a,b} = \mathcal{P}^{m^2a,m^2b}$.
- For the tensor product with a line bundle $\mathcal{L}$ where numerically $ch_1(\mathcal{L}) = H$, since E^n is an abelian variety, the Todd class is trivial, and numerically $ch(\mathcal{L}^{-1}) = e^{-H}$. Thus:

$$Z^{a,b}(v(F\otimes\mathcal{L}^{-1})) = -ch_n^{(b+ia)H}(F\otimes\mathcal{L}^{-1}) = -ch_n^{(b+1+ia)H}(F) = Z^{a,b+1}(v(F))$$

Again, by the uniqueness result in Theorem 2.1, this implies that the stability condition is shifted to $\sigma^{a,b}\otimes\mathcal{L} = \sigma^{a,b+1}$.

□

2.2 Verification of 2 important property

The Bayer property is verified in 1.3, and the following is verification of restrict-N property.

Proposition 2.6 (Proposition 5.4 [Li26])**.** *For any integer* $N \in \mathbb{Z}_{\geq 1}$*, there exists a stability condition* $\sigma^{a,b}$ *on* E^n *as in Theorem 5.2 such that*

$$\sigma^{a,b}\otimes\mathcal{L}^N \leq \sigma^{a,b}[1].$$

Proof. Let $\sigma^{a,b} = (\mathcal{P}^{a,b}, Z^{a,b})$ be a given stability condition on E^n as constructed in Theorem 2.2. By Bridgeland Deformation Theorem, the map forgetting the slicing is a local homeomorphism. Thus, there exists a sufficiently small $\delta_0 > 0$ such that for any rational number δ with $|\delta| \leq \delta_0$, there exists a unique stability condition ω whose central charge is exactly $Z^{a,b+\delta}$.

By shrinking δ_0 if necessary, we can ensure that $\mathrm{dist}(\sigma^{a,b}, \omega) < 1$, which means

$$\sup_{0\neq E\in D^b(E^n)} |\phi^+_{\sigma^{a,b}}(E) - \phi^+_{\omega}(E)| < 1.$$

and it gives $\omega \leq \sigma^{a,b}[1]$ which is equivalent to $\mathcal{P}_\omega(\theta) \subset \mathcal{P}_{\sigma^{a,b}[1]}(<\theta) = \mathcal{P}_{\sigma^{a,b}}(<\theta+1)$, follows directly from inequality above.

Now, choose an integer m sufficiently large such that $\frac{1}{m^2} \leq \delta_0$, setting $\delta = \frac{1}{m^2}$, the corresponding stability condition ω has central charge $Z^{a,b+1/m^2}$, so coincide with $\sigma^{a,b+1/m^2}$ by the uniqueness provided by Theorems 2.1 and 2.2. By explicit computation in Lemma 2.5

$$(\pi_m)_\# \left((\pi_m^\# \mathcal{P}^{a,b}) \otimes \mathcal{L} \right) = \mathcal{P}^{a,b+1/m^2} \leq \mathcal{P}^{a,b}[1]$$

Since $\mathcal{P}^{a,b+1/m^2}$ is the slicing of $\omega = \sigma^{a,b+1/m^2}$.

Next, apply $\pi_m^\#$ to both sides, we obtain:

$$(\pi_m^\# \mathcal{P}^{a,b}) \otimes \mathcal{L} \leq \pi_m^\# \mathcal{P}^{a,b}[1].$$

since $(\pi_m)_\# \pi_m^\# \mathcal{P}^{a,b} = \pi_m^\# (\pi_m)_\# \mathcal{P}^{a,b} = \mathcal{P}^{a,b}$ Using $\pi_m^\# \mathcal{P}^{a,b} = \mathcal{P}^{m^2 a, m^2 b}$, we can rewrite this strictly in terms of stability conditions:

$$\sigma^{m^2 a, m^2 b} \otimes \mathcal{L} \leq \sigma^{m^2 a, m^2 b}[1].$$

Then pull back the inequality by π_k, since $\pi_k^\# \left(\sigma^{m^2 a, m^2 b} \otimes \mathcal{L} \right) = \sigma^{k^2 m^2 a, k^2 m^2 b} \otimes \mathcal{L}^{k^2}$ so we obtain

$$\sigma^{(km)^2 a, (km)^2 b} \otimes \mathcal{L}^{k^2} \leq \sigma^{(km)^2 a, (km)^2 b}[1].$$

Finally, for the given $N \in \mathbb{Z}_{\geq 1}$, we simply choose k large enough such that $k^2 \geq N$. And $\sigma \otimes \mathcal{L}^s \leq \sigma \otimes \mathcal{L}^t$ whenever $s \leq t$ by Lemma 1.2 and Lemma 1.3. Applying this to our case yields:

$$\sigma^{(km)^2 a, (km)^2 b} \otimes \mathcal{L}^N \leq \sigma^{(km)^2 a, (km)^2 b} \otimes \mathcal{L}^{k^2} \leq \sigma^{(km)^2 a, (km)^2 b}[1].$$

So pick $a' = (km)^2 a$ and $b' = (km)^2 b$, the stability condition $\sigma^{a', b'}$ satisfies the Restriction-N property. This completes the proof. □

3 Descending to Arbitrary Projective Varieties

3.1 combinatorial filtration by Double Schubert polynomials

Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$. The symmetric group S_n acts on R by permuting the variables. This section studies a natural filtration on the R-R bimodule $R \otimes_{R^{S_n}} R$ [Li26, Section 4].

For each $w \in S_n$, define the inversion set and the associated polynomial:

$$inv(w) := \{(i, j) \mid 1 \leq i < j \leq n, w(i) > w(j)\},$$

$$\Delta_w := \prod_{(i,j) \in inv(w)} (x_i - x_j)$$

Viewing S_n as a Coxeter group with simple reflections $s_j = (j\ j+1)$ for $1 \leq j \leq n-1$, let $l(w)$ denote the Bruhat length of w. For each $1 \leq j \leq n-1$, the Demazure operator is defined as:

$$\partial_j := \frac{id - s_j}{x_j - x_{j+1}}$$

.

Given a reduced expression $w = s_{a_1} \ldots s_{a_{l(w)}}$, the operator ∂_w is defined as:

$$\partial_w := \partial_{a_1} \ldots \partial_{a_{l(w)}}$$

For the identity element $e \in S_n$, set $\partial_e = id$.

The **Schubert polynomial** associated to $w \in S_n$ is defined by:

$$\mathfrak{S}_w(x) := \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\dots x_{n-1}),$$

where $w_0 = [n, n-1, ..., 2, 1]$ or $w_0(i) = n+1-i$ denotes the longest element of S_n. In fact, the set $\{\mathfrak{S}_w(x)\}_{w \in S_n}$ forms a free basis of R as an R^{S_n}-module.

The **double Schubert polynomial** is defined as:

$$\mathfrak{S}_w(x; y) := \partial_{w^{-1}w_0}\Delta(x; y) = \sum_{\substack{u,v \in S_n; w=v^{-1}u; \\ l(w)=l(v)+l(u)}} \mathfrak{S}_u(x)\mathfrak{S}_v(-y)$$

where $\Delta(x; y) := \prod_{i+j \le n}(x_i - y_j)$.

For any $w' \in S_n$ with $l(w') \le l(w)$, direct computations shows

$$\mathfrak{S}_w(w'x; x) = (-1)^{l(w)}\delta_{w,w'}\Delta_w(x)$$

Let R_w denote the R-bimodule with the usual left R-action and the right R-action twisted by w which is explicitly written as:

$$(R \otimes R) \times R_w \to R_w, \quad (f \otimes g, h) \mapsto fhg(x_{w(1)}, \dots, x_{w(n)})$$

There is a natural R-bimodule homomorphism:

$$F_w : R \otimes_{R^{S_n}} R \longrightarrow R_w,$$

$$f \otimes g \mapsto f \cdot g(x_{w(1)}, \dots, x_{w(n)})$$

Lemma 3.1. *[Li26, Lemma 4.1] The R-bimodule $R \otimes_{R^{S_n}} R$ admits a filtration*

$$0 = \Gamma_{\binom{n}{2}+1} \subset \Gamma_{\binom{n}{2}} \subset \dots \subset \Gamma_1 \subset \Gamma_0 = R \otimes_{R^{S_n}} R$$

by R-bimodules such that, for every j, there is an R-bimodule isomorphism

$$\bigoplus_{l(w)=j} F_w : \Gamma_j/\Gamma_{j+1} \cong \bigoplus_{l(w)=j} \Delta_w \cdot R_w.$$

Let $Y = (\mathbb{P}^1)^n$, the symmetric group S_n acts on Y by permuting the factors, giving a quotient map $f : Y \longrightarrow Y/S_n \cong \mathbb{P}^n$. For each $w \in S_n$, define:

$$\mathcal{L}_w := \mathcal{O}_{\mathbb{P}^1}(a_1) \otimes \dots \otimes \mathcal{O}_{\mathbb{P}^1}(a_n), \quad a_i = \text{the degree of } x_i \text{ in } \Delta_w(x),$$

$$Gr_w := \{(wx, x) \mid x \in Y\} \subset Y \times Y$$

Extending the filtration from 3.1 to sheaves yields the following corollary.

Corollary 3.2. *[Li26, Corollary 4.2] The sheaf $\mathcal{O}_{Y \times_{Y/S_n} Y} \in Coh(Y \times Y)$ admits a filtration*

$$0 = \mathcal{E}_{\binom{n}{2}+1} \subset \mathcal{E}_{\binom{n}{2}} \subset \dots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{O}_{Y \times_{Y/S_n} Y}$$

whose successive quotients are given by

$$\mathcal{E}_j/\mathcal{E}_{j+1} \cong \bigoplus_{w \in S_n; l(w)=j} \mathcal{O}_{Gr_w} \otimes p_2^*\mathcal{L}_w^{-1}.$$

This corollary comes from direct translation of algebraic geometry, one can prove that:

$$\mathcal{O}_{\mathbb{C}^n \times_{\mathbb{C}^n/S_n} \mathbb{C}^n} \cong R \widetilde{\otimes_{R^{S_n}}} R$$

so one can prove the filtration of $\mathcal{O}_{\mathbb{C}^n \times_{\mathbb{C}^n/S_n} \mathbb{C}^n}$ on $\mathbb{C}^n \times \mathbb{C}^n$ by 3.1.

In 3.1, the successive quotients are isomorphic to $\bigoplus_{l(w)=j} \Delta_w \cdot R_w$, which correspond to $\mathcal{O}_{Gr_w \cap (\mathbb{C}^n \times \mathbb{C}^n)}$, we do homogenization to go from $\mathbb{C}^n$ to $\mathbb{P}^n$, recall that $\Delta_w(x)$ is a polynomial where the variable x_i appears with degree a_i, which means $\Delta_w(x)$ globalizes to a section of the line bundle:

$$\mathcal{L}_w := \mathcal{O}_{\mathbb{P}^1}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(a_n) \text{ on } Y$$

and the multiplication by Δ_w introduces a degree shift in the grading. Therefore, the affine piece $\Delta_w \cdot R_w$ naturally globalizes to:

$$\mathcal{O}_{Gr_w} \otimes p_2^* \mathcal{L}_w^{-1}$$

That concludes the proof.

3.2 Inducing stability conditions on projective space

Let $\tau_E : E \to E$ be the reflection involution of the elliptic curve E, and let $\pi_E : E \to E/\langle \tau_E \rangle \cong \mathbb{P}^1$ be the corresponding double cover. For any smooth projective variety M, these morphisms naturally extend to $\tau = (\tau_E, id_M) : E \times M \to E \times M$ and $f = (\pi_E, id_M) : E \times M \to \mathbb{P}^1 \times M$.

Lemma 3.3. *[Li26, Lemma 6.1] Assume that σ is a τ-invariant stability condition on $E \times M$. Then $f_\# \sigma$ is a stability condition on $\mathbb{P}^1 \times M$.*

Proof. We apply Proposition 1.12 with the finite surjective morphism $f : E \times M = Y \to X = \mathbb{P}^1 \times M$. We need to show that $f^* f_* \mathcal{P}_\sigma(\theta) \subset \mathcal{P}_\sigma(\leq \theta)$ for all $\theta \in \mathbb{R}$.

Consider the fiber product $Z = Y \times_X Y$ with natural projections $p_1, p_2 : Z \to Y$. Since f is a double cover branched along the fixed locus of the involution τ, Z has two irreducible components: the diagonal $\Delta(Y)$ and the graph of the involution $\mathrm{Gr}_\tau = \{(\tau y, y) \mid y \in Y\}$. These two components intersect exactly at the branch divisor $D = E^\tau \times M$, where E^τ represents the four 2-torsion points fixed by τ_E.

Then using the exact sequence $0 \to \mathcal{O}_B(-A \cap B) \to \mathcal{O}_{A \cup B} \to \mathcal{O}_A \to 0$ and pick $A = \Delta(Y)$ and $B = \mathrm{Gr}_\tau$, we obtain the exact sequence:

$$0 \longrightarrow \mathcal{O}_{Gr_\tau} \otimes p_2^* \mathcal{O}_Y(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{\Delta(Y)} \longrightarrow 0.$$

For any object $G \in \mathcal{P}_\sigma(\theta)$, we apply the functor $p_{2*}(p_1^*(-) \otimes \bullet)$ to this sequence, we obtain a distinguished triangle in $D^b(Y)$:

$$\tau^* G \otimes \mathcal{O}_Y(-D) \longrightarrow f^* f_* G \longrightarrow G \xrightarrow{+1} .$$

By Lemma 1.3 and 1.2, since σ is τ-invariant, we have:

$$\phi_\sigma^+(\tau^* G \otimes \mathcal{O}_Y(-D)) = \phi^+_{\sigma \otimes \mathcal{O}_Y(D)}(\tau^* G) \leq \phi_\sigma(\tau^* G) = \phi_\sigma(G) = \theta.$$

combining with the distinguished triangle above, we obtain

$$\phi_\sigma^+(f^* f_* G) \leq \max\{\phi_\sigma^+(\tau^* G \otimes \mathcal{O}_Y(-D)), \phi_\sigma(G)\} \leq \theta.$$

This proves $f^* f_* G \in \mathcal{P}_\sigma(\leq \theta)$. By Proposition 1.12, $f_\# \sigma$ is a stability condition on $\mathbb{P}^1 \times M$. □

Proposition 3.4. *[Li26, Proposition 6.2] Let σ be a stability condition on $(\mathbb{P}^1)^n$ satisfying:*

- *σ is S_n-invariant;*
- *$\sigma \leq \sigma \otimes \mathcal{L}$ for every effective line bundle $\mathcal{L}$.*

Then $f_{\#}\sigma$ is a stability condition on $\mathbb{P}^n$.

Proof. We apply the induction criterion from Proposition 1.12 to the finite surjective quotient morphism $f : Y = (\mathbb{P}^1)^n \to X = \mathbb{P}^n$. To do this, we must verify that for any $\theta \in \mathbb{R}$, $f^* f_* \mathcal{P}_\sigma(\theta) \subset \mathcal{P}_\sigma(\leq \theta)$.

Consider the fiber product $Z = Y \times_X Y$ with natural projections $p_1, p_2 : Z \to Y$. For any object $E \in D^b(Y)$, flat base change yields the functorial isomorphism $f^* f_* E \cong p_{2*}(p_1^* E \otimes \mathcal{O}_Z)$ now we use corollary 3.2 to give a finite filtration of $\mathcal{O}_Z$ whose successive quotients are of the form $\mathcal{O}_{Gr_w} \otimes p_2^* \mathcal{L}_w^{-1}$ for each $w \in S_n$, where Gr_ω is the graph of the permutation ω. So $f^* f_* E$ simplify precisely to:

$$\{w^* E \otimes \mathcal{L}_w^{-1}\}_{w \in S_n}.$$

Now, suppose $E \in \mathcal{P}_\sigma(\theta)$. Since σ is S_n-invariant, the pullback $w^* E \in \mathcal{P}_\sigma(\theta)$ for all $w \in S_n$. By our hypothesis, $\sigma \leq \sigma \otimes \mathcal{L}_w$ for the effective line bundle $\mathcal{L}_w$. Using Lemma 1.2, we can bound the maximum phase of the twisted object:

$$\phi_\sigma^+(w^* E \otimes \mathcal{L}_w^{-1}) = \phi^+_{\sigma \otimes \mathcal{L}_w}(w^* E) \leq \phi_\sigma(w^* E) = \theta.$$

Because $f^* f_* E$ is built via extensions of these factors, its maximum HN phase is bounded by the maximum phase of its factors. Therefore, $\phi_\sigma^+(f^* f_* E) \leq \theta$, which means $f^* f_* E \in \mathcal{P}_\sigma(\leq \theta)$, so $f_{\#}\sigma$ is a valid Bridgeland stability condition on $\mathbb{P}^n$ by Proposition 1.12.

This satisfies the condition of Proposition 3.4, concluding that □

Theorem 3.5. *[Li26, Theorem 6.3] Any stability condition $\sigma^{a,b}$ on $(\mathbb{P}^1)^n$ as in Theorem 2.2 induces a stability condition $\pi_{\#}\sigma^{a,b}$ on $\mathbb{P}^n$.*

Proof. The quotient map $\pi : E^n \to \mathbb{P}^n$ by the Weyl group $G = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ naturally factors into two finite surjective morphisms:

$$E^n \xrightarrow{h} E^n/(\mathbb{Z}/2\mathbb{Z})^n \cong (\mathbb{P}^1)^n \xrightarrow{f} (\mathbb{P}^1)^n/S_n \cong \mathbb{P}^n.$$

First, observe that the central charge $Z^{a,b}$ and the lattice Λ of the stability condition $\sigma^{a,b}$ are invariant under the action of G. Since the G-action preserves central charge and the phase of skyscraper sheaf, $\sigma^{a,b}$ itself is globally G-invariant.

Because $\sigma^{a,b}$ is invariant under the reflections $(\mathbb{Z}/2\mathbb{Z})^n$, we can apply Lemma 3.3 iteratively to ensure that $h_{\#}\sigma^{a,b}$ is a well-defined stability condition on $(\mathbb{P}^1)^n$. Furthermore, since the original $\sigma^{a,b}$ was S_n-invariant then $h_{\#}\sigma$ is S_n invariant.

Next, we must verify the second condition of Proposition 3.4 for $h_{\#}\sigma^{a,b}$. Let $\mathcal{L}_i = p_i^* \mathcal{O}_{\mathbb{P}^1}(1)$ be the generator of the Picard group for the i-th factor of $(\mathbb{P}^1)^n$. By Lemma 1.3, $\sigma^{a,b} \leq \sigma^{a,b} \otimes h^* \mathcal{L}_i$.

Applying the pushforward $h_{\#}$ and Lemma 1.11 and 1.10, we obtain:

$$h_{\#}\sigma^{a,b} \leq h_{\#}(\sigma^{a,b} \otimes h^* \mathcal{L}_i) = (h_{\#}\sigma^{a,b}) \otimes \mathcal{L}_i.$$

so

$$h_{\#}\sigma^{a,b} \leq (h_{\#}\sigma^{a,b}) \otimes \mathcal{L}.$$

for any line bundle $\mathcal{L}$ since any effective line bundle $\mathcal{L}$ is a tensor product of positive powers of the $\mathcal{L}_i$'s.

Now, $h_\# \sigma^{a,b}$ satisfies both the S_n-invariance and the tensor inequality required by Proposition 3.4, that concludes the proof. □

Corollary 3.6. *[Li26, Corollary 6.4] For any $N \in \mathbb{Z}_{\geq 1}$, there exist stability conditions $\sigma \in Stab_{K_0(\mathbb{P}^n)}(\mathbb{P}^n)$ satisfying*

$$\sigma \leq \sigma \otimes \mathcal{O}(1) \quad \textit{and} \quad \sigma \otimes \mathcal{O}(N) \leq \sigma[1].$$

Moreover, all skyscraper sheaves are in $\mathcal{P}_\sigma(1)$.

Proof. By Proposition 5.4, we can choose parameters a and b large enough such that the stability condition $\sigma^{a,b}$ on E^n satisfies the restriction-$2N$ property:

$$\sigma^{a,b} \otimes \mathcal{L}^{2N} \leq \sigma^{a,b}[1],$$

where $\mathcal{L}$ is the line bundle on E^n with $\mathrm{ch}_1(\mathcal{L})$ is numerically equivalent to H. By Theorem 3.5, $\sigma := \pi_\# \sigma^{a,b}$ is a stability condition on $\mathbb{P}^n$.

Using lemma 1.3 1.10 and 1.11,

$$\sigma = \pi_\# \sigma^{a,b} \leq \pi_\#(\sigma^{a,b} \otimes \mathcal{L}^2) = \pi_\#(\sigma^{a,b} \otimes \pi^* \mathcal{O}_{\mathbb{P}^n}(1)) = \sigma \otimes \mathcal{O}_{\mathbb{P}^n}(1).$$

here the second equality comes from $\pi^*\mathcal{O}(1)$ is numerically equivalent to $\mathcal{L}^2$ with $\mathrm{ch}_1(\pi^*\mathcal{O}(1)) = 2H$.

$$\sigma \otimes \mathcal{O}_{P^n}(N) = \pi_\#(\sigma^{a,b} \otimes \pi^* \mathcal{O}_{P^n}(N)) = \pi_\#(\sigma^{a,b} \otimes \mathcal{L}^{2N}) \leq \pi_\#(\sigma^{a,b}[1]) = \sigma[1].$$

Finally, we track the skyscraper sheaves. By Theorem 2.2, all skyscraper sheaves on E^n belong to $\mathcal{P}_{\sigma^{a,b}}(1)$. For any closed point $p \in \mathbb{P}^n$, its pullback $\pi^*\mathcal{O}_p$ is a direct sum of skyscraper sheaves on E^n supported on the orbit of the preimage. Since $\mathcal{P}_{\sigma^{a,b}}(1)$ is an additive subcategory, $\pi^*\mathcal{O}_p \in \mathcal{P}_{\sigma^{a,b}}(1)$, this implies $\mathcal{O}_p \in \mathcal{P}_\sigma(1)$. □

3.3 Restricting to arbitrary projective varieties

Theorem 3.7. *[Li26, Theorem 6.5] Let X be a projective variety over $\mathbb{C}$. Then there exists a Bridgeland stability condition on $D^b(X)$.*

Proof. Let $\iota : X \hookrightarrow \mathbb{P}^n$ for some sufficiently large integer n, by the Hilbert Syzygy Theorem, any finitely generated graded module over the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ has a finite free resolution of length at most n. Consequently, the coherent sheaf $\iota_*\mathcal{O}_X \in Coh(\mathbb{P}^n)$ admits a finite locally free resolution of the form:

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n}(-a_{n-1})^{\oplus m_{n-1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-a_1)^{\oplus m_1} \to \mathcal{O}_{\mathbb{P}^n} \to \iota_*\mathcal{O}_X \to 0,$$

for some non-negative integers $a_j, m_j \geq 0$, and where the kernel $\mathcal{F} \in Coh(\mathbb{P}^n)$ is also a coherent sheaf.

Choose an integer $N \gg 0$ large enough such that $jN \geq a_j$ for all $1 \leq j \leq n-1$. By Corollary 3.6, there exists a Bridgeland stability condition σ on $\mathbb{P}^n$ satisfying:

$$\sigma \leq \sigma \otimes \mathcal{O}_{\mathbb{P}^n}(1) \quad \text{and} \quad \sigma \otimes \mathcal{O}_{\mathbb{P}^n}(N) \leq \sigma[1].$$

Using Lemma 1.2,

$$\sigma \otimes \mathcal{O}_{P^n}(a_j) \leq \sigma \otimes \mathcal{O}_{P^n}(jN) \leq \sigma \otimes \mathcal{O}_{P^n}((j-1)N)[1] \leq \cdots \leq \sigma[j].$$

To prove that $\iota^! \sigma$ defines a valid stability condition on X, we apply the induction criterion from Polishchuk [Pol07, Corollary 2.2.2], it suffices toshow for any $E \in \mathcal{P}_\sigma(\theta)$ with $\theta \in (0, 1]$, we have

$$\phi_\sigma^-(\iota_* \mathcal{O}_X \otimes E) \geq \theta$$

by Hilbert Syzygy above, we know

$$\phi_\sigma^-(\iota_* \mathcal{O}_X \otimes E) \geq \min_j \left\{ \phi_\sigma^-(E), \phi_\sigma^-(E \otimes \mathcal{O}_{P^n}(-a_j)[j]), \phi_\sigma^-(E \otimes \mathcal{F}[n]) \right\}.$$

We now estimate each term in this minimum:

1. Obviously, $\phi_\sigma^-(E) = \theta$.
2. Since $\sigma \otimes \mathcal{O}_{P^n}(a_j) \leq \sigma[j]$, which translates via Lemma 1.2 to $\phi_\sigma^-(E \otimes \mathcal{O}_{P^n}(-a_j)[j]) \geq \phi_\sigma^-(E) = \theta$.
3. For the final term $E \otimes \mathcal{F}[n]$, we rely on the homological dimension bounds. Since all skyscraper sheaves $\mathcal{O}_p$ on $\mathbb{P}^n$ belong to $\mathcal{P}_\sigma(1)$, according to [Bri08, Lemma 10.1],

$$E \in \mathcal{P}_\sigma(\theta) \subset \mathrm{Coh}(\mathbb{P}^n)[0, n-1] \quad \mathrm{Coh}(\mathbb{P}^n) \subset \mathcal{P}_\sigma((1-n, 1])$$

then we know $E \otimes \mathcal{F}[n] \in \mathcal{P}_\sigma((1, 3n])$, which ensures that $E \otimes \mathcal{F}[n] \in \mathcal{P}_\sigma((1, 3n])$. an object $E \in \mathcal{P}_\sigma(\theta)$ for $\theta \in (0, 1]$ is a complex concentrated in degrees $[0, n-1]$,

By above 3 conditions, we conclude that:

$$\phi_\sigma^-(\iota_* \mathcal{O}_X \otimes E) \geq \theta.$$

which completes the proof. □

4 Open Problems

- *Topology and Geometry of* $\mathrm{Stab}(X)$*:* Investigate the global structure of the stability manifold. Key questions include determining the number of connected components, identifying non-trivial open subsets, and understanding its detailed topological and manifold properties.

  A small result in [WZ26] gives the non-empty property of G-invariant stability conditions, which is quite easy to prove.

  Proposition 4.1. *Let X be a smooth projective variety with an action of a finite group G. Then $\mathrm{Stab}(X)^G$ is non-empty.*

- *Progress of the Γ-BMT Conjecture:* Apply the newly established techniques and existence results from Li (2026) to attack and potentially solve the Γ-BMT conjecture for higher-dimensional varieties.

  Conjecture 4.2 (BMT-conjecture)**.** *[BMT14, conjecture 4.1]*

  For X smooth projective threefold, and $H \in NS(X)$ an ample class. Assume that E is $\nu^H_{\alpha,\beta}$-semistable then:

$$\alpha^2 \overline{\Delta}_H(E) + 4(H \,\mathrm{ch}_2^\beta(E))^2 - 6H^2 \,\mathrm{ch}_1^\beta(E) \,\mathrm{ch}_3^\beta(E) \geq 0$$

  equivalently, for any tilt-semistable object $E \in \mathcal{B}_{\omega,B}$ satisfying $\nu_{\omega,B}(E) = 0$,

$$ch_3^B(E) \leq \frac{\omega^2}{18} ch_1^B(E).$$

  here $B = \beta H$ and $\omega = \sqrt{3}\alpha H$ and $\overline{\Delta}_H(E) = (H^2 ch_1^B(E))^2 - 2H^3 ch_0^B(E) H ch_2^B(E)$

Conjecture 4.3 (Γ-BMT Conjecture, [**?**])**.** *Let X be a smooth projective threefold and H an ample divisor on X, E be a $\nu_{\beta,\alpha}$-semistable object. Find a 1-cycle $\Gamma \in A_1(X)_{\mathbb{R}}$ satisfying $\Gamma.H \geq 0$, then $Q^{\Gamma}_{\alpha,\beta}(E) \geq 0$ holds.*

Here, the quadratic form $Q^{\Gamma}_{\alpha,\beta}$ is defined as:

$$\begin{aligned} Q^{\Gamma}_{\alpha,\beta}(E) := (2\alpha - \beta^2)\left(\overline{\Delta}_H(E) + 3\frac{\Gamma.H}{H^3}(H^3 ch_0^{\beta}(E))^2\right) \\ + 2(Hch_2^{\beta}(E))(2Hch_2^{\beta}(E) - 3\Gamma.Hch_0^{\beta}(E)) \\ - 6(H^2ch_1^{\beta}(E))(ch_3^{\beta}(E) - \Gamma ch_1^{\beta}(E)) \end{aligned}$$

Remark 4.4. The primary motivation for introducing BMT-conjecture in [BMS16] is to establish the support property for Bridgeland stability conditions on threefolds, such as Abelian threefolds.

Here are the results of Feyzbakhsh of Γ-BMT conjecture for threefolds.

Theorem 4.5 (Feyzbakhsh, Koseki, Liu, Rekuski, 2025)**.** *Let (X, H) be a polarised Calabi-Yau threefold. The Γ-BMT conjecture holds for (X, H) and some 1-cycle Γ with $\Gamma.H \geq 0$ if it belongs to one of the following classes:*

1. *X is a quasi-smooth complete intersection Calabi-Yau threefold in a weighted projective space and H is the generator of $Pic(X)$.*
2. *X is a general divisor in $|-K_M|$ of a Fano fourfold M of index r and $H = (-\frac{1}{r}K_M)|_X$, where either $r \geq 3$ or both $r = 2$ and M has Picard number one.*
3. *X is a cyclic cover $\pi : X \to Y$ of a Fano threefold Y of index r and $H = \pi^*(-\frac{1}{r}K_Y)$, where either $r \geq 2$ or both $r = 1$ and Y has Picard number one.*

Remark 4.6 (Proof Strategy: Dimensional Reduction)**.** The proof idea relies on a powerful three-step dimensional reduction strategy, effectively converting the highly non-linear ch_3 inequality on a threefold into a Brill-Noether problem on a curve.

And Feyzbakhsh will give a big progress Γ-BMT conjecture soon.

- *Complete Classification for Special Varieties:* Classify all Bridgeland stability conditions on abelian varieties or specific products of elliptic curves, such as E^n endowed with complex multiplication (CM).

Definition 4.7. An elliptic curve $E \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ lattice in $\mathbb{C}$ is said to be *without complex multiplication* (or generic) if

$$\mathrm{End}(E) \cong \mathbb{Z}.$$

In this generic case, the only analytic scalings that preserve the lattice Λ are real integers.

And E is said to *have complex multiplication (CM)* if its endomorphism ring is strictly larger than $\mathbb{Z}$, in this case

$$\mathrm{End}(E) \cong \mathcal{O} \subset \mathbb{Q}(\sqrt{-d})$$

for some integer $d > 0$, for example, if $\Lambda = \mathbb{Z} \oplus \mathbb{Z}i$, it has complex multiplication.

Theorem 4.8 (Rank of the Numerical Grothendieck Group)**.** *Let E be an elliptic curve and E^n be its n-fold product.*

– *If E is a generic elliptic curve without CM, the rank is given by:*

$$\mathrm{rk}\, K_{\mathrm{num}}(D^b(E^n)) = \frac{1}{n+2}\binom{2n+2}{n+1} = 2, 5, 14, 42, \ldots$$

– *If E is an elliptic curve with CM, the rank is given by:*

$$\operatorname{rk} K_{\text{num}}(D^b(E^n)) = \binom{2n}{n} = 2, 6, 20, 70, \ldots$$

Consequently, the CM case achieves the maximal rank among all abelian varieties of dimension n, which implies that the stability manifold $\operatorname{Stab}(E^n)$ attains the maximal possible dimension.

Conjecture 4.9 (Haiden–Sung)**.** *Let E be any elliptic curve with complex multiplication (e.g., $E = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$). The space of Bridgeland stability conditions on its bounded derived category is exactly the universal cover of the maximal domain:*

$$\operatorname{Stab}(D^b(E^n)) = \widetilde{\mathcal{V}^+(n)}.$$

Furthermore, for a general abelian variety A, the space $\operatorname{Stab}(D^b(A))$ is expected to be the universal cover of a linear slice of $\mathcal{V}^+(n)$ of dimension $\operatorname{rk} K_{\text{num}}(A)$.

Here we need to define $\widetilde{\mathcal{V}^+(n)}$ completely and in detail, which comes from [Hai18].

Definition 4.10 (The classifying domain $\mathcal{V}^+(n)$ and its universal cover)**.**

Let (W, ω) be a symplectic vector space of dimension $2n$ over $\mathbb{R}$. We define the classifying domain $\mathcal{V}^+(n)$ and its universal cover $\widetilde{\mathcal{V}^+(n)}$ through the following steps[cite: 8]:

– Let $\wedge^n_{\text{pr}} W^\vee$ denote the space of primitive n-forms on W. An n-form α is primitive if $\alpha \wedge \omega = 0$

– Define $\mathcal{V}(W)$ as the set of complex-valued primitive n-forms $\Omega \in (\wedge^n_{\text{pr}} W^\vee) \otimes \mathbb{C}$ that are non-vanishing on all Lagrangian subspaces $L \subset W$.

– Any form $\Omega \in \mathcal{V}(W)$ induces a phase map $\phi_\Omega : \operatorname{LGr}(W) \to \mathbb{R}/\pi\mathbb{Z}$ given by $L \mapsto \operatorname{Arg}(\Omega|_L)$ since $\wedge^n_{\text{pr}} L^\vee \otimes \mathbb{C} \cong \mathbb{C}$ for any $L \in \operatorname{LGr}(W)$, where $\operatorname{LGr}(W)$ is the Lagrangian Grassmannian.

– Define $\mathcal{V}^+(W)$ as the connected component of $\mathcal{V}(W)$ consisting of forms Ω for which the induced map ϕ_Ω on fundamental groups sends the canonical generator of $\pi_1(\operatorname{LGr}(W)) \cong \mathbb{Z}$ to a positive loop in $\pi_1(\mathbb{R}/\pi\mathbb{Z})$.

– Finally, define $\widetilde{\mathcal{V}^+(n)}$ as the universal covering space of $\mathcal{V}^+(W)$ (where $W \cong \mathbb{R}^{2n}$).

Remark 4.11. In their upcoming joint work, Haiden and B. Sung prove this conjecture specifically for the case $n = 3$. We already know the case when $n = 3$ (means $\operatorname{Stab}(D^b(E^n))$) for E with or without complex multiplication. But for $n \geq 4$, the proof technique of $n = 3$ completely fails, and proving $n = 4$ now facing a big difficulty.

- *Stability conditions on special categories such as $\mathcal{H}iggs(X)$:* Develop stability conditions and related theoretical frameworks for specialized categories except $D^b(\operatorname{Coh}(X))$, such as $\mathcal{H}iggs(X)$.

Definition 4.12 (The Category of Higgs Sheaves $\mathcal{H}iggs(X)$)**.**

Let X be a smooth projective variety over $\mathbb{C}$, and let ω_X be its canonical line bundle. The category of Higgs sheaves on X, denoted by $\mathcal{H}iggs(X)$, is defined as follows:

– **Objects:** An object is a pair (E, θ), where $E \in \operatorname{Coh}(X)$ is a coherent sheaf on X, and θ is a morphism of coherent sheaves called the *Higgs field*:

$$\theta : E \to E \otimes \Omega^1_X$$

which satisfies $\theta \wedge \theta = 0$. For a curve, $\Omega_X^1 = \omega_X$ and the integrability condition is automatically satisfied.

- **Morphisms:** A morphism between two Higgs sheaves (E_1, θ_1) and (E_2, θ_2) is a morphism of coherent sheaves $f : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\theta_1} & E_1 \otimes \Omega_X^1 \\ \downarrow f & & \downarrow f \otimes \mathrm{id}_{\Omega_X^1} \\ E_2 & \xrightarrow{\theta_2} & E_2 \otimes \Omega_X^1 \end{array}$$

  In other words, f must satisfy $(f \otimes \mathrm{id}_{\Omega_X^1}) \circ \theta_1 = \theta_2 \circ f$.

The category $\mathcal{H}iggs(X)$ is an abelian category.

Remark 4.13. A good perspective of research is developing the theory of Bridgeland stability conditions for Higgs sheaves / Higgs bundles, or constructing a parallel dictionary generalizing the classical theory on $\mathrm{Coh}(X)$ to $\mathcal{H}iggs(X)$. Here are some possible structural steps:

- Give precise construction of $D^b(\mathcal{H}iggs(X))$ and explore basic properties.
- Defining "slope stability" on $D^b(\mathcal{H}iggs(X))$, here one can change subsheaves to θ-invariant subsheaves $0 \subsetneq F \subsetneq E$, i.e., $\theta(F) \subset F \otimes \omega_X$.
- Gives "tilting" and "t-structures" on $D^b(\mathcal{H}iggs(X))$, comparing with [BMS16, Section 2] construction.
- Gives central charges and Bridgeland stability, here we must focus on support properties, and exactly defines $\mathrm{Stab}(D^b(\mathcal{H}iggs(X)))$, determine whether it is a manifold or an algebraic variety.
- Research on moduli spaces, wall-crossing behavior and general properties such as connected component or non-empty of $\mathrm{Stab}(D^b(\mathcal{H}iggs(X)))$, for some simple X, try to compute $\mathrm{Stab}(D^b(\mathcal{H}iggs(X)))$.

- *Stability conditions on singular varieties*

  Here one can try to compute the precise total stability space of X with singularities, such as $\dim X = 1$ with ADE type singularity, or Abelian surface with some simple singularities, but it is very hard to compute, but we can change $D^b(\mathrm{Coh}(X))$ to $D_{sg}(X)$, which has some good properties.

Definition 4.14 (Singularity Category [Orl04])**.**

Let X be a Noetherian algebraic scheme. The derived category of perfect complexes, denoted by $\mathrm{Perf}(X)$, is the full triangulated subcategory of $D^b(\mathrm{Coh}(X))$ consisting of complexes that are locally quasi-isomorphic to bounded complexes of locally free sheaves, and $D_{sg}(X)$ is defined as:

$$D_{sg}(X) := D^b(\mathrm{Coh}(X))/\,\mathrm{Perf}(X).$$

For a smooth scheme X, every coherent sheaf admits a finite locally free resolution, means $D_{sg}(X) \cong 0$, therefore, $D_{sg}(X)$ isolates and captures the “purely singular” homological phenomena of X.

Theorem 4.15 (Bridgeland, [Bri09])**.** *Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup, and Γ be the associated Dynkin graph via the McKay correspondence. Let $\mathcal{D}$ be the triangulated category defined as the full subcategory of $\mathcal{D}^b(Coh_G(\mathbb{C}^2))$ consisting of complexes whose cohomology sheaves are supported at the origin and have no non-trivial G-invariant sections.*

Let $\mathfrak{g}$ be the finite-dimensional complex simple Lie algebra corresponding to the Dynkin graph Γ, and let $\mathfrak{h} \subset \mathfrak{g}$ be its Cartan subalgebra. Let $\mathfrak{h}^{reg} \subset \mathfrak{h}$ be the regular semisimple part defined as

$$\mathfrak{h}^{reg} = \{\theta \in \mathfrak{h} : \alpha(\theta) \neq 0 \text{ for all roots } \alpha \in \Lambda\}.$$

Let W be the Weyl group acting freely on $\mathfrak{h}^{reg}$.

Then, there exists a connected component of the space of Bridgeland stability conditions, denoted by $Stab_0(\mathcal{D}) \subset Stab(\mathcal{D})$, which is a regular covering space of the quotient space $\mathfrak{h}^{reg}/W$.

Furthermore, the subgroup $Br(\mathcal{D}) \subset Aut(\mathcal{D})$, generated by the twist functors (spherical twists) along the simple objects of the canonical heart, preserves this connected component $Stab_0(\mathcal{D})$ and acts as the group of deck transformations.

Remark 4.16. The fundamental group of the quotient space $\mathfrak{h}^{reg}/W$ is naturally isomorphic to the Artin braid group $\mathrm{Br}(\Gamma)$ associated to the Dynkin graph Γ. Bridgeland's theorem provides a surjective group homomorphism from the braid group to the group of deck transformations:

$$\rho : \mathrm{Br}(\Gamma) \to \mathrm{Br}(\mathcal{D}).$$

In the special case of type A_n, relying on the prior results of Seidel and Thomas, ρ is an isomorphism and the covering space $Stab_0(\mathcal{D})$ is simply-connected.

Therefore, for type A_n, $Stab_0(\mathcal{D})$ is precisely the universal cover of $\mathfrak{h}^{reg}/W$. For general ADE types, showing that $Stab_0(\mathcal{D})$ is the universal cover requires independently proving that $Stab_0(\mathcal{D})$ is simply-connected, which Bridgeland posed as a natural hope.

References

[BBMT13] A. Bayer, A. Bertram, E. Macrì, and Y. Toda. *Bridgeland stability conditions on threefolds II: An application to Fujita's conjecture.* arXiv:1106.3430v2 [math.AG], 2013.

[BMS16] A. Bayer, E. Macrì, and P. Stellari. *The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds.* Invent. Math., 206(3):869–933, 2016.

[BMT14] A. Bayer, E. Macrì, and Y. Toda. *Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities.* J. Algebraic Geom., 23(1):117–163, 2014.

[Bri08] T. Bridgeland. *Stability conditions on K3 surfaces.* Duke Math. J., 141(2):241–291, 2008.

[Bri09] T. Bridgeland, *Stability conditions and Kleinian singularities*, International Mathematics Research Notices (IMRN), 2009(21), 4142–4157, 2009.

[BS16] B. Schmidt. *Counterexample to the generalized Bogomolov-Gieseker inequality for threefolds.* arXiv:1602.05055v1 [math.AG], 2016.

[Hai18] F. Haiden, *An extension of the Siegel space of complex abelian varieties and conjectures on stability structures*, arXiv:1808.06364 (2018).

[KQ15] Y. Qiu and A. King, *Exchange graphs and Ext quivers*, Advances in Mathematics, 285, 1106–1154, 2015.

[Li25] C. Li. *A real reduction of the manifold of Bridgeland stability conditions.* arXiv:2506.21995, 2025.

[Li26] C. Li. *A Remark on Stability Conditions on Smooth Projective Varieties.* arXiv:2601.22994 [math.AG], 2026.

[Orl04] D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Proceedings of the Steklov Institute of Mathematics, 246(3), 227–248, 2004.

[Pol07] A. Polishchuk. *Constant families of t-structures on derived categories of coherent sheaves*. Mosc. Math. J., 7(1):109–134, 2007.

[WZ26] D. Wu and N. Zhang, *The G-noncommutative minimal model program*, arXiv preprint arXiv:2602.20335v1 [math.AG], 2026.