Hypersurface with Automorphism Group of Large Order

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2026.3

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1 Basic Definitions and Facts

[1] Fix an algebraically closed field $k = \overline{k}$. Let $n \geq 1$, $d \geq 3$ be integers. Let[2] $F \in k[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d. We say F if smooth if $F = \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0$ has no solution in $\mathbb{P}^n_k$.

Definition 1.1. We write X_F, $\mathrm{Aut}(F)$ and $\mathrm{Lin}(X_F)$ for the hypersurface of dimension $n-2$ in $\mathbb{P}^n_k$, the subgroup of $\mathrm{GL}(n,k)$ preserving F, and the linear part[3] of automorphism group of X_F *resp.*

Theorem 1.2 ([MM63],[Cha78],e.t.c.)**.** *If $n \geq 3$ and $(n,d) \neq (3,3),(4,4)$, then* $\mathrm{Aut}(X_F)$ *equals*[4] $\mathrm{Lin}(X_F)$ *and is finite.*

It has been long wondered for fixed (n,d), what is the maximal size of $\mathrm{Lin}(X_F)$? A brief review of the history can be found in the introduction of [YYZ25].

Definition 1.3. Let $\rho : G \to \mathrm{GL}(n,k)$ be a irreducible representation of group G. We say ρ is primitive if k^n cannot be decomposed to proper subspaces such that G acts by permutations on those subspaces. The definition of irreduciblity and this definition canonically applies to subgroup $G < \mathrm{GL}(n,k)$. For subgroup $G < \mathrm{PGL}(n,k)$, we say it is primitive if $G = \pi\left(\widetilde{G}\right)$ for some primitive $\widetilde{G} < \mathrm{GL}(n,k)$.

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[1] This is the note for the second talk at 2026 Spring Learning Seminar. There might many typos and lazy statements.

[2] People usually use $n+2$ in place of n, because if so n equals the dimension of the hypersurface.

[3] Here the notion is a bit abused: $\mathrm{Lin}(X_F)$ does not only depend on X_F, it actually presents an invariant just for F.

[4] I think, in the problem we are considering, this equality is not so important.

Lemma 1.4 ([Col08, Lemma 1]). *If $G < \mathrm{GL}(n, \mathbb{C})$ is a primitive subgroup, then any normal abelian subgroup of G is cyclic and central.*

Proof. Let $V = \mathbb{C}^n$ be the module. Let N be an abelian normal subgroup of G. Suppose, N is either non-cyclic or cyclic but not central in G.

Since N is abelian, the restriction $V|_N$, as an N-module, decomposes into a direct sum of submodules corresponding to characters of N:

$$V|_N = \bigoplus_{\chi} V_\chi,$$

where $V_\chi = \{v \in V \mid n \cdot v = \chi(n)v \text{ for all } n \in N\}$ and χ runs over some set of characters of N.

We claim that under our assumption, $V|_N$ is **inhomogeneous**[5]. That is, there exist at least two distinct characters $\chi_1 \neq \chi_2$ such that $V_{\chi_1} \neq 0$ and $V_{\chi_2} \neq 0$.

If $V|_N$ is homogeneous, N acts faithfully by scalar on it, which implies N is cyclic.

Moreover, since N is normal in G, $\chi^g(n) = \chi(g^{-1}ng)$ will also occur in the decomposition of $V|_N$. Since $V|_N$ is faithuful, we have $N < Z(G)$.

Therefore, $V|_N$ is inhomogeneous. Therefore G acts by permutation on $\{V_\lambda : V_\lambda \neq 0\}$, which contradicts to G is primitive.

□

Generally for a subgroup $G < \mathrm{GL}(n, \mathbb{C})$, we can decompose $V = \mathbb{C}^n$ into irreducible subrepresentations as $V = \bigoplus_{i=1}^m V_i$. Furthermore, each V_i can be decomposed to $V_i = \bigoplus_{j=1}^{k_i} W_{ij}$ such that G acts by permutation on $W_{i1}, \dots, W_{ik_i}$, namely by $G \twoheadrightarrow K_i < S_{k_i}$, and $\mathrm{Stab}_G(W_{ij})$ acts primitively on W_{ij}. Write $r_{ij} = \dim W_{ij}$ and $H_{ij} = \mathbb{P}(\mathrm{Stab}_G(W_{ij})|_{W_{ij}}) < \mathrm{PGL}(W_{ij})$.

2 Addictive Decomposition

[6] Let[7] $R = k[x_1, \dots, x_n]$ and R_d be the space of degree d homogeneous polynomials in R. I will briefly describe the uniqueness of addictive multiplication.

By $\mathrm{Ann}(F)$ we mean the k-linear space of elements in $k[\partial_1, \dots, \partial_n]$ annihilating F, which is graded naturally.

Definition 2.1. For $F \in R_d$, let M_F be the set of matrices

$$M_F := \{A \in M_{n\times n}(k) : I_2([\partial\ A\partial]) \subset \mathrm{Ann}(f)\}$$

where $\partial = (\partial_1, \dots, \partial_n)^t$, $[\partial\ A\partial]$ is a $r \times 2$ matrix and I_2 is the ideal generated by its 2×2 minors.

Definition 2.2. Let $\gamma_F : M_F \to R_d$ be the map sending A to the unique $G \in R_d$ such that $\partial G = A\partial G$, which is k-linear.

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[5] Let G be a group and V a representation of G. We say that V is **homogeneous** if, when decomposed into irreducible subrepresentations, all irreducible components are isomorphic to each other.

[6] This part was not discussed on the seminar. Only the result of uniqueness of decomposition is roughly mentioned.

[7] For simplicity we assume k is algebriacally closed of characteristic zero, for example $\mathbb{C}$. I am not really sure about how those results goes when $\mathrm{char}\, k > 0$.

An important property is

Proposition 2.3. *If* $d \geq 3$*,* $F \in R_d$*, then* M_F *is a* k*-algebra with all commutators belonging to* $\ker \gamma_F$*. Moreover* M_F *is commutative if* $\operatorname{Ann}(F)_1 = 0$*.*

Note that $\operatorname{Ann}(F)_1 = 0$ is equivalent to say F has no "dummy" variables. In the cases we care about we can always assume this condition holds.

Definition 2.4. For $G_i \in R_d, 1 \leq i \leq n$, we say they are **in independent variables** if

$$R_{d-1}(G_i) \cap \left(\sum_{j \neq i} R_{d-1}(G_j) \right) = 0, \forall i.$$

It is natural to think of $R_{d-1}(f)$ as naive variables of $f \in R_d$.

Definition 2.5. For $f \in R_d$, we say f **splits regularly** $m - 1$ times if f is a sum of m non-zero forms of degree d in independent variables.

Theorem 2.6. *If* $d \geq 2$ *and* $\operatorname{Ann}(F)_1 = 0$*, let* $\operatorname{Coid}(M_F)$ *be the set of all coids* $\{e_1, \ldots, e_n\}$ *of orthogonal idempotents in* M_f*, and let*

$$\operatorname{Reg}(F) = \{\{G_1, \ldots, G_n\} : f = G_1 + \cdots + G_n \text{ is a regular splitting of } f\}.$$

Then the map $\{E_i\}_i \mapsto \{G_i = \gamma_F(E_i)\}_i$ *defines a bijection* $\operatorname{Coid}(M_F) \xrightarrow{\sim} \operatorname{Reg}(F)$*. Moreover, when* $d \geq 3$ *there is a unique maximal regular splitting.*

Proposition 2.7. *Assume* $F = F(x_1, \ldots, x_n)$ *is smooth of degree* $d \geq 3$ *and has maximal regular splitting*

$$F = \sum_{1 \leq i \leq m, 1 \leq j \leq k_i} G_{ij}$$

where for fixed i*,* G_{ij}*'s are isomorphic (say, for conveniences, to* G_i*). Then*

$$\operatorname{Aut}(F) \cong \prod_i \left((\operatorname{Aut}(G_i))^{k_i} \rtimes S_{k_i} \right)$$

has order $\prod_i \left(k_i! |\operatorname{Aut}(G_i)|^{k_i} \right)$*.*

3 Some Bounds

Theorem 3.1. *For* $n \geq 1$*, then any finite subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *has a normal abelian subgroup of index not larger than some constant depending only on* n*. Denote the optimal bound by* $J(n)$*, and the optimal bound for primitive subgroups by* $JC(n)$*. Moreover,* $J(n) = (n+1)!$ *for* $n \geq 71$*, and* $JC(n) = (n+1)!$ *for* $n \geq 13$*.*

Theorem 3.2. *Assume* $F = F(x_1, \ldots, x_n)$ *be smooth of degree* $d \geq 3$*,* $n \geq 2$*, fix a partition* $n = n_1 + \cdots + n_k$*, then* $|\operatorname{Aut}(F) \cap \{\operatorname{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_k I_{n_k}\}| \leq d^k$*.*

Proof. We may assume $n \geq 3$. Let $T < \mathrm{PGL}(n, \mathbb{C})$ be the group of diagonal matrices. Consider a point $x \in X := X_F \subset \mathbb{P}^{n-1}$ with non-zero coordinates, then $\operatorname{Stab}_T(x)$ is trivial and we can identify sets T and Tx. Let $U = X \cap Tx$, then $\operatorname{Lin}(X) \cap T = \bigcap_{s\in T, sx\in U} Us^{-1}$.

Let $H = \mathbb{P}(\{\operatorname{diag}(\lambda_1 I_{n_1}, \dots, \lambda_k I_{n_k}\}) < T$, and Y be the closure of Hx in $\mathbb{P}^{n-1}$, then $Y \cong \mathbb{P}^{k-1}$. We have $\operatorname{Lin}(X) \cap H = \operatorname{Lin}(X) \cap T \cap H = \bigcap_{s\in T, sx\in U}(Us^{-1} \cap H) \subset \bigcap_{s\in T, sx\in U}(Us^{-1} \cap Y) = \operatorname{Lin}(X) \cap T \cap Y$, which is a finite set. The finiteness tells that for generic $s_1, \dots, s_{k-1}$, we have already $\bigcap_j (U{s_j}^{-1} \cap Y)$ being finite.

By Bézout's theorem we have $|\operatorname{Lin}(X) \cap H| \le d^{k-1}$, which implies the conclusion. □

Lemma 3.3. *If $F = F(x_1, \dots, x_n)$ be smooth of degree $d \ge 3$ and $n \ge 2$. If $\operatorname{Aut}(F)$ consists of semi-permutations, then $|\operatorname{Aut}(F)| \le n!d^n$, and the equality holds only if F is isomorphic to the (n, d)-Fermat polynomial.*

4 Estimation

Following notations in Proposition 2.7, we can assume G_i is in n_i variables. Combining with Theorems 3.1 and 3.2, we are to estimate the proportion (divided by the order of the automorphism group of the Fermat polynomial)

$$\frac{\prod_i \left(k_i!|\operatorname{Aut}(G_i)|^{k_i}\right)}{n!d^n} \le d^{k_1+\cdots+k_m}\frac{\prod_i \left(k_i!JC(n_i)^{k_i}\right)}{n!d^n}.$$

Denote LHS(RHS *resp.*) in the above inequality by $L(F)$($R(F)$ *resp.*). We want to show, for large n, $L(F) \le 1$ and the equality is achieved by the Fermat polynomial.

Consider decomposition $l = (n_1^{k_1} \cdots n_m^{k_m})$ of n i.e. $1 \le n_1 < \cdots < n_m$, and $n = \sum_i k_i n_i$. Define $v(l) = n$ backwards, and the length of l being $\sum_i k_i$. We can define partial orders on the set of decomposition of positive integers, by $(n_1^{k_1} \cdots n_m^{k_m}) \ge (n_1^{k_1'} \cdots n_m^{k_m'})$ if $k_i \ge k_i'$. We can define addition and difference between decompositions as well.

Define the corresponding ratio by[8]

$$R(l, d) = d^{k_1+\cdots+k_m}\frac{\prod_i \left(k_i!JC(n_i)^{k_i}\right)}{n!d^n}, d \ge 3.$$

which is non-decreasing in d.

Lemma 4.1. *a) $\binom{v(l_1+l_2)}{v(l_1)} \ge \frac{R(l_1,d)R(l_2,d)}{R(l_1+l_2,d)}$. The equality hods iff factors in l_1 and l_2 are disjoint.*

b) If factors in l_1 and l_2 are disjoint, $v(l_2') \le v(l_2)$ but $R(l_2', d) \ge R(l_2, d)$, then $R(l_1 + l_2', d) \ge R(l_1 + l_2, d)$.

c) Suppose the length of l_1 is not smaller than that of l_2, $v(l_1) \le v(l_2)$. If $R(l_1, d) > R(l_2, d)$, then $R(l_1, d + 1) > R(l_2, d + 1)$.

Lemma 4.2. *Suppose l contains $r_0 > 1$ with multiplicity $k_0 \ge 1$. Then for $d \ge 3$, $R(l, d) > R(l + (r_0^1), d)$ holds if one of the following conditions is true:*

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[8] I should warn and apologize for the notation here, that at the beginning I stupidly think unpartitionable polynomials must have primitive automorphism group. That is, anyway, obviouly wrong, generic polynomial would be a counterexample. So k_i's and n_i's do not coincide in the two side, and maybe the theory of regular splittings does not really provide any simplifications for the proof.

i) $r_0 = 2$ *and* $k_0 \geq 5$*;*

ii) $r_0 = 4$ *and* $k_0 \geq 2$*;*

iii) $r_0 = 3$ *or* $r_0 \geq 5$*.*

Moreover, if $l = (r_0^{k_0})$ *with* $r_0 \notin \{2, 3, 4, 5, 6, 8\}$*, then* $R(l, d) < 1$*.*

Lemma 4.3. *Let* $d \geq 3$ *and* $l = (r_1^{k_1} \cdots r_m^{k_m})$ *with* $R(r_i^{k_i}, d) \geq 1$ *for all* i*. Then* $R(l, d) \leq R(l', d)$ *, where*

$$l' = \begin{cases} (4^1), & \text{if } v(l) = 4 \text{ and } d = 3 \\ \left(2^{\left[\frac{v(l)}{2}\right]}\right). & \text{if } (v(l), d) \neq (4, 3) \end{cases}$$

Moreover, generally if $v(l) \geq 28$*, we have* $R(l, d) < 1$*.*

Theorem 4.4. *Let* $F = F(x_1, \ldots, x_n)$ *be smooth form of degree* $d \geq 3$*. Suppose that* X_F *is not isomorphic to the Fermat hypersurface. If* $n \geq 26$ *or* $d \geq 18$*, then* $|\operatorname{Aut}(X_F)| < n!d^{n-1}$*.*

5 Some Questions

Question 5.1. What if k is algebraically closed of characteristic $p > 0$?

Question 5.2. Fix (n, d), what is the maximum of $|\operatorname{Aut}(F)|$ among smooth unpartitionable/purely non-Fermat F?

For above questions one can at least consider the cases n, d or p are sufficiently large.

Conjecture 5.3. [9] *Fix* (n, d)*, when* $\operatorname{char} k$ *is sufficiently large, possibilities of* $\operatorname{Aut}(F)$ *as well as* $\operatorname{Lin}(X_F)$ *are the same as the characteristic zero case.*

Conjecture 5.4. *For any fixed* k, d *and sufficiently large* n*, if* $F \in \mathbb{C}[x_1, \ldots, x_n]_d$ *is smooth and purely non-Fermat, then* $|\operatorname{Aut}(F)| < d^{n-k}(n-k)!$*.*

Example 5.5. If d is not prime, let $d = qd'$ where q is the minimal prime divisor of d, then

$$F = \sum_i x_i^d + (-1)^{q-1} \left(\sum_i x_i^{d'} \right)^q$$

is unpartitionable(in most cases) with automorphism group of size at least ${d'}^n n!$

Proposition 5.6. *If Conjecture 5.4 holds, then for fixed* m, d*, when* n *is sufficiently large, the maximal to the* m*-th maximal smooth* (n, d)*-polynomials are just ones with as large as possible Fermat part.*

References

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[9] I am not so sure whether this conjecture is obviously wrong or obviously correct.

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