MIXED HODGE THEORY FOR UNITARY LOCAL SYSTEMS

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1. Introduction to Mixed Hodge Structures

We all know the famous Hodge decomposition:

Theorem 1.1 (Hodge Theorem for Kähler Manifolds)**.** Suppose X is a compact Kähler manifold. Then we have a decomposition

$$H^k_{dR}(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X).$$

Moreover, for each p, q, complex conjugate induces a $\bar{\mathbb{C}}$-linear isomorphism $H^{p,q}_{\bar{\partial}}(X) \to H^{q,p}_{\bar{\partial}}(X)$.

This theorem motivates the following definition of abstract Hodge structures.

Definition. Suppose V is a finite dimensional $\mathbb{R}$-vector space. A real Hodge structure on V is a direct sum decomposition of $\mathbb{C}$-vector spaces

$$V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q},$$

with $V^{q,p} = \overline{V^{p,q}}$.

If R is a subring of $\mathbb{R}$, and V_R is a finitely generated free R-module, such that $V \cong V_R \otimes_R \mathbb{R}$, then we say V_R carries an R-Hodge structure.

A morphism of R-Hodge structures is an R-linear map $V_R \to W_R$ such that its complexification sends $V^{p,q}$ to $W^{p,q}$.

Given a real Hodge structure on V, denote

$$V^{(k)} = \bigoplus_{p+q=k} V^{p,q},$$

called the weight k part of V (note it is not a subspace of V!). If $V_{\mathbb{C}} = V^{(k)}$, then we say V carries a weight k real Hodge structure. If moreover $V = V_R \otimes_R \mathbb{R}$, we say V_R carries a weight k R-Hodge structure. If $R = \mathbb{Z}$, we often omit $\mathbb{Z}$ and say $V_{\mathbb{Z}}$ carries a weight k Hodge structure.

If V carries a weight k real Hodge structure, we define the Hodge filtration to be

$$F^p V_{\mathbb{C}} = \bigoplus_{r\geq p} V^{p,q}.$$

It is a decreasing filtration on $V_\mathbb{C}$ satisfying $V_\mathbb{C} = F^p \oplus \overline{F^q}$ whenever $p+q=k+1$. Conversely, any decreasing filtration F^p on $V_\mathbb{C}$ satisfying this property defines a weight k real Hodge structure on V via $V^{p,q} = F^p \cap \overline{F^q}$, for all $p+q=k$.

Suppose V, W are real vector spaces with Hodge structures of weight k, l, respectively. Then we can define a Hodge filtration on $(V \otimes_\mathbb{R} W)_\mathbb{C}$ by

$$F^p(V \otimes_\mathbb{R} W)_\mathbb{C} = \sum_m F^m(V_\mathbb{C}) \otimes_\mathbb{C} F^{p-m}(W_\mathbb{C}) \subset V_\mathbb{C} \otimes_\mathbb{C} W_\mathbb{C}.$$

This defines a Hodge structure of weight $k+l$ on $V \otimes_\mathbb{R} W$. The Hodge filtration

$$F^p(\mathrm{Hom}_\mathbb{R}(V,W)_\mathbb{C}) = \{f : V_\mathbb{C} \to W_\mathbb{C} | f(F^n(V_\mathbb{C})) \subset F^{n+p}(W_\mathbb{C})\}$$

defines a Hodge structure of weight $l-k$ on $\mathrm{Hom}_\mathbb{R}(V,W)$.

One naturally asks what happens for other geometric objects, for example, quasi-projective varieties. Then one obtains a so-called mixed Hodge structure.

Let R be a noetherian subring of $\mathbb{C}$, such that $R \otimes_\mathbb{Z} \mathbb{Q}$ is a field (which must be $\mathrm{Frac}\, R$). Let V_R be a finite free R-module.

Definition. An R-mixed Hodge structure on V_R consists of two filtrations, an increasing filtration $W_\bullet$ on $V_{R\otimes_\mathbb{Z}\mathbb{Q}} = V_R \otimes_\mathbb{Z} \mathbb{Q}$, called the weight filtration, and a decreasing filtration $F^\bullet$ on $V_\mathbb{C} = V_R \otimes_R \mathbb{C}$, called the Hodge filtration, such that $F^\bullet$ induces a pure Hodge structure of weight k on each $\mathrm{Gr}_k^W(V_{R\otimes_\mathbb{Z}\mathbb{Q}}) = W_k/W_{k-1}$, i.e. $F^p(W_k/W_{k-1})_\mathbb{C} := \frac{F^p \cap (W_k)_\mathbb{C} + (W_{k-1})_\mathbb{C}}{(W_{k-1})_\mathbb{C}}$ is a Hodge filtration of weight k on $(W_k/W_{k-1})_\mathbb{C}$.

2. Mixed Hodge Structures Arising from Quasi-projective Varieties

In this section, we explain how cohomologies of quasi-projective varieties carry a mixed Hodge structure.

2.1. Simple Normal Crossing Divisors and Logarithmic Poles.

Suppose X is a smooth complex algebraic variety.

Definition. A normal crossing divisor is a divisor that locally looks like union of coordinate hyperplanes (and each component has multiplicity 1). A simple normal crossing divisor is a normal crossing divisor with smooth components.

Let D be a simple normal crossing divisor, and $U = X \backslash D$.

Definition. A holomorphic differential form on U is said to have logarithmic poles along D if $\omega, d\omega$ have at most poles of order 1 along D.

The holomorphic differential forms on U having logarithmic poles along D constitute a subcomplex $\Omega_X^\bullet(\log D) \subset j_*\Omega_U^\bullet$.

If locally D is defined by $z_1 \dots z_k = 0$, with $(z_1, \dots, z_n)$ a system of local coordinates, then locally $\Omega_X^1(\log D) = \mathcal{O}_X \frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_X \frac{dz_k}{z_k} \oplus \mathcal{O}_X dz_{k+1} \oplus \cdots \oplus \mathcal{O}_X dz_n$, and $\Omega_X^k(\log D) = \bigwedge^k \Omega_X^1(\log D)$.

2.1.1. *The two filtrations on the logarithmic complex.* The weight filtration on $\Omega_X^\bullet(\log D)$ is defined by

$$W_m \Omega_X^p(\log D) = \begin{cases} \Omega_X^m(\log D) \wedge \Omega_X^{p-m}, & m < p \\ \Omega_X^p(\log D), & m \geq p. \end{cases}$$

The Hodge filtration on $\Omega_X^\bullet(\log D)$ is equal to the trivial filtration, that is,

$$F^p = (0 \to \cdots \to 0 \to \Omega_X^p(\log D) \to \Omega_X^{p+1}(\log D) \to \cdots).$$

2.2. Some Quasi-isomorphisms of Complexes.

Theorem 2.1. The followings are quasi-isomorphic (i.e. isomorphic in the derived category):

$$\Omega_X^\bullet(\log D), j_*\Omega_U^\bullet, Rj_*\mathbb{C}_U, Rj_*\Omega_U^\bullet.$$

Proof. $\Omega_U^\bullet$ is a resolution of $\mathbb{C}_U$, hence $Rj_*\mathbb{C}_U \simeq Rj_*\Omega_U^\bullet$. By Proposition 3.12 in Audoubert and Tommasi's note [AT], $\Omega_U^\bullet$ is j_*-acyclic, thus $Rj_*\Omega_U^\bullet \simeq j_*\Omega_U^\bullet$. By Theorem 3.13 in Audoubert and Tommasi's note [AT], we have a quasi-isomorphism $\Omega_X^\bullet(\log D) \to j_*\Omega_U^\bullet$. □

Now we have

Lemma 2.2.
$$\mathbb{H}^i(X,\Omega_X^\bullet(\log D))\cong H^i(U,\mathbb{C}_U).$$

Proof. We have
$$\begin{aligned}\mathbb{H}^k(X,\Omega_X^\bullet(\log D)) &\cong \mathbb{H}^k(X,j_*\Omega_U^\bullet)\\ &\cong h^k(R\Gamma(j_*\Omega_U^\bullet))\\ &\cong h^k(R\Gamma(Rj_*\Omega_U^\bullet))\\ &\cong h^k(R(\Gamma\circ j_*)\Omega_U^\bullet))\\ &= h^k(R\Gamma(\Omega_U^\bullet))\\ &\cong H^k(U,\mathbb{C}_U).\end{aligned}$$
□

So we can define mixed Hodge structures on $H^k(U,\mathbb{C}_U)$ by weight and Hodge filtrations on the complex $\Omega_X^\bullet(\log D))$.

Suppose $D=\sum D_i$. For a set I of indices, denote $D_I=\cap_{i\in I}D_i$ (as a regular submanifold of X). Denote
$$D_m=\bigcup_{|I|=m}D_I, D^{[m]}=\bigsqcup_{|I|=m}D_I.$$
Denote $a_m:D^{[m]}\to X$ the canonical map. It is proper with finite fibers.

By the holomorphic Poincare lemma, $\Omega^\bullet_{D^{[m]}}$ is a resolution for $\mathbb{C}_{D^{[m]}}$, and hence $(a_m)_*\Omega^\bullet_{D^{[m]}}$ is a resolution for $(a_m)_*\mathbb{C}_{D^{[m]}}$. So

Lemma 2.3. The followings are quasi-isomorphic:
$$(a_m)_*\mathbb{C}_{D^{[m]}}(0),(a_m)_*\Omega^\bullet_{D^{[m]}}.$$

If $|I|=m$, then we have a residue map
$$\mathrm{Res}_I:\Omega_X^\bullet(\log D)\to\Omega^\bullet_{D_I}(\log D(I))[-m],$$
defined by (if $I=\{i_1,\dots i_m\}$) $\frac{dz_{i_1}}{z_{i_1}}\wedge\dots\wedge\frac{dz_{i_m}}{z_{i_m}}\wedge\omega+\omega'\mapsto\omega|_{D_I}$, where ω' is not divisible by $\frac{dz_{i_1}}{z_{i_1}}\wedge\dots\wedge\frac{dz_{i_m}}{z_{i_m}}$, and $D(I)=\sum_{j\notin I}D_I\cap D_j$.

Remark. Here a choice of the orders of $i_1,\dots,i_m$, is involved. A different choice of the orders can lead to a sign change of Res_I. So to be more rigorous, we introduce the rank 1 local system $\epsilon^I=\bigwedge^m\mathbb{C}^I$ on D_I. Then Res_I is in fact a morphism
$$\Omega_X^\bullet(\log D)\to\Omega^\bullet_{D_I}(\log D(I))\otimes_{\mathbb{C}}\epsilon^I[-m].$$

We also denote ϵ^m to be the disjoint union of these local systems on $D^{[m]}$. Note that in the case D being a simple normal crossing divisor, the local system ϵ^m is trivial, so we can ignore it. The Res_I's add up to a morphism
$$\mathrm{Res}_m:\Omega_X^\bullet(\log D)\to\Omega^\bullet_{D^{[m]}}(\log D(m))\otimes_{\mathbb{C}}\epsilon^m[-m].$$
Here $D(m)$ is the divisor on $D^{[m]}$ obtained by pulling back D_{m+1} through $a_m:D^{[m]}\to X$.

Lemma 2.4. The Res_I's induce an isomorphism of complexes of sheaves
$$\mathrm{Res}_m:\mathrm{Gr}_m^W\Omega_X^\bullet(\log D)\to(a_m)_*\Omega^\bullet_{D^{[m]}}[-m].$$

Proof. See [PS08] Lemma 4.6. □

So the following are quasi-isomorphic:
$$\mathrm{Gr}_m^W\Omega_X^\bullet(\log D),(a_m)_*\Omega^\bullet_{D^{[m]}}[-m],(a_m)_*\mathbb{C}_{D^{[m]}}[-m]$$

This allows us to compute the cohomology sheaves of $\mathrm{Gr}_m^W\Omega_X^\bullet(\log D)$:
$$h^i(\mathrm{Gr}_m^W\Omega_X^\bullet(\log D))\cong h^i((a_m)_*\mathbb{C}_{D^{[m]}}[-m])=\begin{cases}(a_m)_*\mathbb{C}_{D^{[m]}}, & i=m,\\ 0, & i\neq m.\end{cases}$$

By induction on m, one has

$$h^i(W_m\Omega_X^\bullet(\log D)) \cong \begin{cases}(a_i)_*\mathbb{C}_{D^{[i]}}, & 0\le i\le m,\\ 0, & i>m,\end{cases}$$

where $0 \le m \le \dim_{\mathbb{C}} X$. In particular, take $m = \dim_{\mathbb{C}} X$, we get

$$h^i(\Omega_X^\bullet(\log D)) \cong (a_i)_*\mathbb{C}_{D^{[i]}}, 0 \le i \le \dim_{\mathbb{C}} X.$$

2.3. The Hodge–de Rham Complex of (X, D).

Definition. Let $(K^\bullet, d_K)$ be a bounded below differential complex of objects in some abelian category.

- The *trivial filtration* on $K^\bullet$ is the decreasing filtration $\sigma^{\ge}$,

$$\sigma^{\ge p} := \{0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K^p \longrightarrow K^{p+1} \longrightarrow \dots\}.$$

- The *canonical filtration* on $K^\bullet$ is the increasing filtration $\tau_{\le}$,

$$\tau_{\le p} := \{K^0 \longrightarrow K^1 \longrightarrow \cdots \longrightarrow K^{p-1} \longrightarrow \mathrm{ker} d_K^p \longrightarrow \dots\}.$$

The associated graded is

$$\mathrm{Gr}_p^{\tau_\le}(K^\bullet) = \{0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K^{p-1}/\mathrm{ker} d_K^{p-1} \longrightarrow \mathrm{ker} d_K^p \longrightarrow \dots\}.$$

There is a canonical map

$$\mathrm{Gr}_p^{\tau_\le}(K^\bullet) \longrightarrow h^p(K^\bullet)[-p],$$

which is a quasi-isomorphism. Consequently,

$$h^p(\mathrm{Gr}_p^{\tau_\le}(K^\bullet)) \cong h^p(K^\bullet).$$

If $K^\bullet \to L^\bullet$ is a quasi-isomorphism, then the filtered morphism $(K^\bullet, \tau_\le) \to (L^\bullet, \tau_\le)$ is a *filtered quasi-isomorphism*, i.e.

$$\mathrm{Gr}_p^{\tau_\le}(K^\bullet) \longrightarrow \mathrm{Gr}_p^{\tau_\le}(L^\bullet)$$

is a quasi-isomorphism for every p.

We apply this to the logarithmic de Rham complex. The inclusion

$$(\Omega_X^\bullet(\log D), \tau_\le) \hookrightarrow (\Omega_X^\bullet(\log D), W)$$

is a filtered morphism, because by definition $W_m\Omega_X^p(\log D) = \Omega_X^p(\log D)$ for $m \ge p$. In fact it is a filtered quasi-isomorphism, since $\mathrm{Gr}_m^{\tau_\le}(\Omega_X^\bullet(\log D))$ has cohomology only in degree m and

$$h^m(\mathrm{Gr}_m^{\tau_\le}(\Omega_X^\bullet(\log D))) \cong h^m(\Omega_X^\bullet(\log D)) \cong h^m(\mathrm{Gr}_m^W\Omega_X^\bullet(\log D)).$$

Moreover,

$$\Omega_X^\bullet(\log D) \xrightarrow{\text{q.is.}} j_*\Omega_{X^*}^\bullet \implies (\Omega_X^\bullet(\log D), \tau_\le) \xrightarrow{\text{q.is.}} (j_*\Omega_{X^*}^\bullet, \tau_\le),$$

and

$$(j_*\Omega_{X^*}^\bullet, \tau_\le) \xrightarrow{\text{q.is.}} (\mathcal{R}^\bullet j_*\Omega_{X^*}^\bullet, \tau_\le) \xrightarrow{\text{q.is.}} (\mathcal{R}^\bullet j_*\mathbb{C}_{X^*}, \tau_\le).$$

Definition. Let X be a compact complex manifold with Kähler metric, and $D \subset X$ a divisor with normal crossings. The *Hodge-de Rham complex* $\mathcal{K}_{DR}^\bullet(X \log D)$ associated to the couple (X, D) is the triple:

(1) $\mathcal{R}^\bullet j_*\mathbb{Z}_{X^*} := j_*\mathcal{C}_{\mathrm{cdm}}^\bullet(\mathbb{Z}_{X^*})$;
(2) $(\mathcal{R}^\bullet j_*\mathbb{Q}_{X^*}, \tau_\le)$ with the natural map

$$\alpha : \mathcal{R}^\bullet j_*\mathbb{Z}_{X^*} \otimes_{\mathbb{Z}_X} \mathbb{Q}_X \xrightarrow{\text{q.is.}} \mathcal{R}^\bullet j_*\mathbb{Q}_{X^*},$$

which is a quasi-isomorphism of complexes of sheaves;
(3) $(\Omega_X^\bullet(\log D), W, F)$ with $F = \sigma^{\ge}$, and the following chain β of filtered quasi-isomorphisms:

$$\begin{array}{ccccc}
 & & & & (Rj_*\mathbb{Q}_U, \tau_\le) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \\
 & & & & \downarrow \\
(\Omega_X^\bullet(\log D), \tau_\le) & & & & (Rj_*\mathbb{C}_U, \tau_\le) \\
\downarrow & \searrow & & & \downarrow \\
(\Omega_X^\bullet(\log D), W) & & (j_*\Omega_U^\bullet, \tau_\le) & \longrightarrow & (Rj_*\Omega_U^\bullet, \tau_\le)
\end{array}$$

Verdier, following Grothendieck, writes (β) as

$$\beta : (\mathcal{R}^\bullet j_*\mathbb{Q}_{X^*}, \tau_\leq) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \xrightarrow{\sim} (\Omega^\bullet_X(\log D), W),$$

considering β as an isomorphism in the derived category of bounded below filtered complexes of sheaves of complex vector spaces (it is not in general a map of sheaves).

Remark. There is a strong analogy between the definition of the Hodge-de Rham complex and the general definition of a mixed Hodge structure: (1) is analogous to the choice of the lattice, (2) to the rational weight filtration, and (3) to the Hodge filtration. In fact, $\mathcal{K}^\bullet_{DR}(X \log D)$ is what is called a *mixed Hodge complex of sheaves*.

All maps considered in $\mathcal{K}^\bullet_{DR}(X \log D)$ are filtered morphisms. Hence they induce morphisms on the associated graded, preserving the same properties. In this way we obtain for every m the triple

$$\big(\mathrm{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_*\mathbb{Z}_{X^*},\ (\mathrm{Gr}^W_m \Omega^\bullet_X(\log D), F),\ \mathrm{Gr}_m(\beta)\big),$$

where

$$\mathrm{Gr}_m(\beta) : \mathrm{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_*\mathbb{Z}_{X^*} \otimes_{\mathbb{Z}_X} \mathbb{C}_X \longrightarrow \mathrm{Gr}^W_m \Omega^\bullet_X(\log D).$$

Now consider the action of the Poincaré residue map on this triple. We have

$$\mathcal{H}^m\big(\mathrm{Gr}^{\tau_\leq}_m \mathcal{R}^\bullet j_*\mathbb{C}_{X^*}\big) = \mathcal{H}^m(\mathcal{R}^\bullet j_*\mathbb{C}_{X^*}) = \mathcal{R}^m j_*\mathbb{C}_{X^*}.$$

We already know

$$\mathcal{R}^m j_*\mathbb{C}_{X^*} = \mathbb{H}^m(X, \Omega^\bullet_X(\log D)) = \mathbb{H}^m(X, \mathrm{Gr}^W_m \Omega^\bullet_X(\log D)) \xrightarrow[\text{q.i.s.}]{\mathrm{Res}_m} (a_m)_*\mathbb{C}_{D^{[m]}}.$$

At the integer level we have:

$$\begin{array}{ccccccc}
R^m j_*\mathbb{C}_U & = & h^m(\Omega^\bullet_X(\log D)) & = & h^m(\mathrm{Gr}^W_m \Omega^\bullet_X(\log D)) & \xrightarrow{\mathrm{Res}_m} & (a_m)_*\mathbb{C}_{D^{[m]}} \\
\uparrow & & & & & & \uparrow \\
R^m j_*\mathbb{Z}_U & & & \xrightarrow[\cong]{\qquad\qquad} & & & (\frac{1}{2\pi i})^m (a_m)_*\mathbb{Z}_{D^{[m]}}
\end{array}$$

The latter sheaf is the $(-m)$-th Tate twist of $(a_m)_*\mathbb{Z}_{D^{[m]}}$:

$$(a_m)_*\mathbb{Z}_{D^{[m]}}(-m) := \left(\frac{1}{2\pi i}\right)^m (a_m)_*\mathbb{Z}_{D^{[m]}}.$$

We now verify the commutativity of the diagram locally. Choose coordinates $(z_1, \dots, z_n)$ satisfying the usual condition $(*)$ near a point $x \in D$. For $m = 1$:

$$\big(\mathcal{R}^1 j_*\mathbb{C}_{X^*}\big)_x = H^1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{C}) = \mathbb{C}\frac{\mathrm{d}z_1}{z_1} \oplus \cdots \oplus \mathbb{C}\frac{\mathrm{d}z_k}{z_k},$$

$$\big(\mathcal{R}^1 j_*\mathbb{Z}_{X^*}\big)_x = H^1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{Z}).$$

The first homology group is

$$H_1((\Delta^*)^k \times \Delta^{n-k}, \mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_k,$$

where γ_j is a loop around the origin in $\Delta_j = \{|z_j| < \epsilon\}$. By the residue formula,

$$\int_{\gamma_j} \frac{\mathrm{d}z_j}{z_j} = 2\pi i.$$

Hence the classes $\{\frac{1}{2\pi i}\frac{\mathrm{d}z_j}{z_j}\}_{j=1,\dots,k}$ represent integral cohomology classes; they form the dual basis over $\mathbb{Z}$ of $\gamma_1, \dots, \gamma_k$. This proves the claim for $m = 1$.

For general m the argument is analogous. The Poincaré residue map induces a quasi-isomorphism

$$\mathrm{Res}_m : \mathrm{Gr}^{\tau_\leq}_m(\mathcal{R}^\bullet j_*\mathbb{Z}_{X^*}) \longrightarrow (a_m)_*\mathbb{Z}_{D^{[m]}}[-m](-m).$$

Consequently we obtain an isomorphism

$$\mathrm{Res}_m : (\mathrm{Gr}^W_m \Omega^\bullet_X(\log D), F) \xrightarrow{\sim} \big((a_m)_*\Omega^\bullet_{D^{[m]}}[-m],\ F[-m]\big).$$

Moreover, Res_m also gives a map corresponding to $\mathrm{Gr}_m(\beta)$, because

$$(a_m)_*\mathbb{Z}_{D^{[m]}}[-m](-m) \otimes_{\mathbb{Z}_X} \mathbb{C}_X \cong (a_m)_*\mathbb{C}_{D^{[m]}}[-m] \xrightarrow{\mathrm{Res}_m} (a_m)_*\Omega^\bullet_{D^{[m]}}[-m].$$

Summarizing, the Poincaré residue map gives a map of triples

$$\big(\mathrm{Gr}_m^{\tau_\leq}\mathcal{R}^\bullet j_*\mathbb{Z}_{X^*},\ (\mathrm{Gr}_m^W\Omega_X^\bullet(\log D),F),\ \mathrm{Gr}_m(\beta)\big)$$
$$\xrightarrow{\mathrm{Res}_m}$$
$$\big((a_m)_*\mathbb{Z}_{D^{[m]}}[-m](-m),\ ((a_m)_*\Omega^\bullet_{D^{[m]}}[-m],F[-m]),\ \mathbb{C}_{D^{[m]}}[0]\xrightarrow{\text{q.i.s.}}\Omega^\bullet_{D^{[m]}}\big),$$

consisting of maps that are isomorphisms in the derived category.

2.4. **The Theory of Deligne.** The general theory of Deligne deals with a complex $K^\bullet$ endowed with two filtrations W and F. It considers the spectral sequence associated to the filtered complex $(K^\bullet, W)$ and investigates how the filtration F induces a filtration there. This study reveals abstract properties ensuring that various hypercohomology groups of a mixed Hodge complex of sheaves yield a mixed Hodge structure. A general principle is then as follows: In order to get a mixed Hodge structure on hypercohomology of some geometric object, it suffices to construct a suitable mixed Hodge complex of sheaves on this object.

2.4.1. *Integral Hodge complex of sheaves.* We will see now that the structures naturally arisen in the previous section are special cases of more general concepts.

Definition ([AT], [PS08])**.** A $\mathbb{Z}$-Hodge complex of weight m is the given of the following:

(1) A bounded below complex of $\mathbb{Z}$-modules $K^\bullet_\mathbb{Z}$ such that $\operatorname{rank} H^k(K^\bullet_\mathbb{Z}) < \infty$ for all k.
(2) A bounded below complex $K^\bullet_\mathbb{C}$ of $\mathbb{C}$-vector spaces equipped with a decreasing filtration F and a comparison morphism
$$\alpha : K^\bullet_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{C}\xrightarrow{\sim}K^\bullet_\mathbb{C},$$
which is an isomorphism in the derived category.

The data $(K^\bullet_\mathbb{Z},(K^\bullet_\mathbb{C},F))$ have moreover to satisfy the following condition, called Hodge completion axiom:

(HC) (i) $\forall k\geq 0$, $(H^k(K^\bullet_\mathbb{Z}),(H^k(K^\bullet_\mathbb{C}),F))$ is an integral Hodge structure of weight $k+m$.
(ii) the spectral sequence of $(K^\bullet_\mathbb{C},F)$ degenerates at E_1 (equivalently, the differential of the complex $K^\bullet_\mathbb{C}$ is strictly compatible with the filtration F).

Definition ([AT], [PS08])**.** A $\mathbb{Z}$-Hodge complex of sheaves on X of weight m, where X is a topological space, is the given of the following:

(1) A bounded below complex $K^\bullet_\mathbb{Z}$ of sheaves of $\mathbb{Z}$-modules on X, such that $\operatorname{rank}\mathbb{H}^k(X,K^\bullet_\mathbb{Z})<\infty$ for all k.
(2) A bounded below complex $K^\bullet_\mathbb{C}$ of sheaves of $\mathbb{C}$-vector spaces, equipped with a decreasing filtration F and a comparison morphism
$$\alpha : K^\bullet_\mathbb{Z}\otimes_\mathbb{Z}\mathbb{C}_X\xrightarrow{\sim}K^\bullet_\mathbb{C},$$
which is an isomorphism in the derived category.

The data $(K^\bullet_\mathbb{Z},(K^\bullet_\mathbb{C},F))$ have to satisfy the following condition (Hodge completion axiom for complexes of sheaves):

(HCS) (i) $(\mathbb{H}^k(X,K^\bullet_\mathbb{Z}),(\mathbb{H}^k(X,K^\bullet_\mathbb{C}),F))$ is an integral Hodge structure of weight $k+m$.
(ii) the spectral sequence of $(\mathcal{R}\Gamma(X,K^\bullet_\mathbb{C}),F)$ degenerates at E_1.

Remark. There is a natural relation between the two definitions:
$$(K^\bullet_\mathbb{Z},(K^\bullet_\mathbb{C},F),\alpha)\xrightarrow{\text{functor}}(\mathcal{R}\Gamma(X,K^\bullet_\mathbb{Z}),(\mathcal{R}\Gamma(X,K^\bullet_\mathbb{C}),F),\mathcal{R}\Gamma(X,\alpha))$$
because, by definition, $\mathbb{H}^k(X,K^\bullet_\mathbb{Z})=H^k\mathcal{R}\Gamma(X,K^\bullet_\mathbb{Z})$, $K^\bullet_\mathbb{C}\xrightarrow{\sim}\mathcal{C}^\bullet_{\text{cdm}}(K^\bullet_\mathbb{C})$ and $\mathcal{R}\Gamma(X,K^\bullet_\mathbb{C})=\Gamma(X,\mathcal{C}^\bullet_{\text{cdm}}(K^\bullet_\mathbb{C}))$. Here $\Gamma(X,\mathcal{C}^\bullet_{\text{cdm}}(-))$ is an exact functor. Then $\mathcal{R}\Gamma(X,K^\bullet_\mathbb{C})$ is filtered by $\mathcal{R}\Gamma(X,F^pK^\bullet_\mathbb{C})$.

Remark. As in the case of mixed Hodge structures, we can define $\mathbb{Q}$- (respectively, $\mathbb{R}$- or $\mathbb{C}$-) Hodge complexes and complexes of sheaves by substituting $\mathbb{Z}$ in the definition with $\mathbb{Q}$ (respectively, $\mathbb{R}$ or $\mathbb{C}$).

Example. Let X be a complex compact Kähler manifold. The Hodge-de Rham complex of sheaves on X is defined by
$$\mathcal{K}^\bullet_{DR}(X)=(\mathbb{Z}_X[0],(\Omega^\bullet_X,\sigma^{\geq}),\mathbb{C}_X[0]\xrightarrow{\sim}\Omega^\bullet_X).$$

(The inclusion $\mathbb{C}_X[0] \hookrightarrow \Omega_X^\bullet$ is a quasi-isomorphism by the holomorphic Poincaré lemma.) $\mathcal{K}^\bullet_{DR}(X)$ is a $\mathbb{Z}$-Hodge complex of sheaves of weight 0. Let us verify the axiom (HCS).

We have to prove that the filtration

$$F^p\mathbb{H}^k(X,\Omega_X^\bullet) := \operatorname{im}\big(\mathbb{H}^k(X,\sigma^{\geq p}\Omega_X^\bullet) \longrightarrow \mathbb{H}^k(X,\Omega_X^\bullet)\big)$$

is a Hodge filtration on $\mathbb{H}^k(X,\Omega_X^\bullet)$. Consider the spectral sequence of $(\mathcal{R}\Gamma(X,\Omega_X^\bullet),\sigma^{\geq})$:

$$E_1^{p,q} = \mathbb{H}^{p+q}(X,\operatorname{Gr}_\sigma^{\sigma^{\geq}}\Omega_X^\bullet) = \mathbb{H}^{p+q}(X,\Omega_X^p[p]) = H^q(X,\Omega_X^p) \cong H^{p,q}(X),$$

$$E_2^{p,q} = \operatorname{Gr}_p^F\mathbb{H}^{p+q}(X,\Omega_X^\bullet) = \operatorname{Gr}_p^F H_{DR}^{p+q}(X,\mathbb{C}).$$

As a consequence,

$$\sum_{p+q=k}\dim E_1^{p,q} = \sum_{p+q=k}\dim H^{p,q}(X) = \dim_{\mathbb{C}} H^k_{DR}(X,\mathbb{C}) = \sum_{p+q=k}\dim E_2^{p,q}.$$

This implies that E_1 and E_∞ have the same dimension; hence the spectral sequence degenerates at E_1. This is equivalent to the map $\mathbb{H}^k(X,\sigma^{\geq p}\Omega_X^\bullet) \to \mathbb{H}^k(X,\Omega_X^\bullet)$ being injective $\forall k$. (See Lemma 4.13 below.)

Proposition 2.5. Let $(K^\bullet, F, \mathrm{d})$ be a bounded below complex with differential $\mathrm{d} : K^\bullet \to K^{\bullet+1}$ and a decreasing filtration F. The following are equivalent:

1. the spectral sequence degenerates at E_1;
2. the map $H^k(F^pK^\bullet) \to H^k(K^\bullet)$ is injective for all k;
3. d is strictly compatible with the filtration F.

Definition. Let $\mathcal{K}^\bullet = (\mathcal{K}^\bullet_{\mathbb{Z}}, (\mathcal{K}^\bullet_{\mathbb{C}}, F), \alpha)$ be a $\mathbb{Z}$-Hodge complex of sheaves of weight m on X. Then for all $n \in \mathbb{Z}$ we define the n-th Tate twist of $\mathcal{K}^\bullet$,

$$\mathcal{K}^\bullet(n) = (\mathcal{K}^\bullet_{\mathbb{Z}}(n), (\mathcal{K}^\bullet_{\mathbb{C}}, F[n]), \alpha)$$

by taking $\mathcal{K}^\bullet_{\mathbb{Z}}(n) := (2\pi i)^n\mathcal{K}^\bullet_{\mathbb{Z}}$ and the usual shifted filtration $(F[n])^p = F^{p+n}$. The effect of the Tate twist on the hypercohomology is

$$\mathbb{H}^k(X, K^\bullet_{\mathbb{C}}(n)) = \mathbb{H}^k(X, K^\bullet_{\mathbb{C}})(n) = \mathbb{H}^k(X, K^\bullet_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathbb{C}(n),$$

where $\mathbb{C}(n) := (2\pi i)^n\mathbb{C}$ is the Tate-Hodge structure on $\mathbb{C}$ of weight $-2n$. Thus $\mathcal{K}^\bullet(n)$ is a $\mathbb{Z}$-Hodge complex of sheaves of weight $m-2n$.

Definition. Let $\mathcal{K}^\bullet = (\mathcal{K}^\bullet_{\mathbb{Z}}, (\mathcal{K}^\bullet_{\mathbb{C}}, F), \alpha)$ be a $\mathbb{Z}$-Hodge complex of sheaves of weight m on X. Then for all $r \in \mathbb{Z}$ we define the shift of $\mathcal{K}^\bullet$ by r as

$$\mathcal{K}^\bullet[r] := (\mathcal{K}^\bullet_{\mathbb{Z}}[r], (\mathcal{K}^\bullet_{\mathbb{C}}[r], F), \alpha).$$

In this case $\mathbb{H}^k(X, \mathcal{K}^\bullet_{\mathbb{Z}}[r]) = \mathbb{H}^{k+r}(X, \mathcal{K}^\bullet_{\mathbb{Z}})$ and $\mathcal{K}^\bullet[r]$ is a $\mathbb{Z}$-Hodge complex of sheaves of weight $m+r$.

2.4.2. *Integral mixed Hodge complex of sheaves.*

Definition ([AT], [PS08])**.** A mixed $\mathbb{Z}$-Hodge complex is the given of the following:

1. A bounded below complex of $\mathbb{Z}$-modules $K^\bullet_{\mathbb{Z}}$, with $\operatorname{rank} H^k(K^\bullet_{\mathbb{Z}}) < \infty$.
2. A bounded below complex $K^\bullet_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces equipped with an increasing filtration W and a comparison morphism
$$\alpha : K^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} K^\bullet_{\mathbb{Q}},$$
which is an isomorphism in the derived category.
3. A bounded below complex $K^\bullet_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces equipped with an increasing filtration W, a decreasing filtration F and a comparison morphism
$$\beta : (K^\bullet_{\mathbb{Q}}, W) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} (K^\bullet_{\mathbb{C}}, W),$$
which is a map in the derived category such that for all m the induced map
$$\operatorname{Gr}^W_m(\beta) : \operatorname{Gr}^W_m(K^\bullet_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \operatorname{Gr}^W_m(K^\bullet_{\mathbb{C}})$$
is an isomorphism in the derived category.

The data $(K^\bullet_{\mathbb{Z}}, (K^\bullet_{\mathbb{Q}}, W), \alpha, (K^\bullet_{\mathbb{C}}, W, F), \beta)$ have moreover to satisfy the following condition:

$$(*) \quad \forall m, \quad (\mathrm{Gr}^W_m(K^\bullet_{\mathbb{Q}}), (\mathrm{Gr}^W_m(K^\bullet_{\mathbb{C}}), F), \mathrm{Gr}^W_m(\beta))$$

is a $\mathbb{Q}$-Hodge complex of weight m.

Definition ([AT], [PS08])**.** A mixed $\mathbb{Z}$-Hodge complex of sheaves on a topological space X is the given of the following:

(1) A bounded below complex $K^\bullet_{\mathbb{Z}}$ of sheaves of $\mathbb{Z}$-modules on X, such that $\operatorname{rank} \mathbb{H}^k(X, K^\bullet_{\mathbb{Z}}) < \infty$.
(2) A bounded below complex $K^\bullet_{\mathbb{Q}}$ of sheaves of $\mathbb{Q}$-vector spaces on X equipped with an increasing filtration $W_\bullet$ and a comparison morphism

$$\alpha : K^\bullet_{\mathbb{Z}} \otimes_{\mathbb{Z}_X} \mathbb{Q}_X \xrightarrow{\sim} K^\bullet_{\mathbb{Q}},$$

which is an isomorphism in the derived category.
(3) A bounded below complex $K^\bullet_{\mathbb{C}}$ of sheaves of $\mathbb{C}$-vector spaces on X equipped with an increasing filtration W, a decreasing filtration F and a comparison morphism

$$\beta : (K^\bullet_{\mathbb{Q}}, W) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \xrightarrow{\sim} (K^\bullet_{\mathbb{C}}, W),$$

which is a map in the derived category such that for all m the induced map

$$\mathrm{Gr}^W_m(\beta) : \mathrm{Gr}^W_m(K^\bullet_{\mathbb{Q}}) \otimes_{\mathbb{Q}_X} \mathbb{C}_X \xrightarrow{\sim} \mathrm{Gr}^W_m(K^\bullet_{\mathbb{C}})$$

is an isomorphism in the derived category.

The data $(\mathcal{K}^\bullet_{\mathbb{Z}}, (\mathcal{K}^\bullet_{\mathbb{Q}}, W), \alpha, (\mathcal{K}^\bullet_{\mathbb{C}}, W, F), \beta)$ have moreover to satisfy the following condition:

$$(**) \quad \forall m, \quad (\mathrm{Gr}^W_m(K^\bullet_{\mathbb{Q}}), (\mathrm{Gr}^W_m(K^\bullet_{\mathbb{C}}), F), \mathrm{Gr}^W_m(\beta))$$

is a $\mathbb{Q}$-Hodge complex of sheaves of weight m.

Example. Let X be a compact Kähler manifold, and $D \subset X$ a normal crossing divisor. Then the Hodge-de Rham complex associated to (X, D) is

$$\mathcal{K}^\bullet_{DR}(X \log D) = (\mathcal{R}^\bullet j_* \mathbb{Z}_{X^*}, (\mathcal{R}^\bullet j_* \mathbb{Q}_{X^*}, \tau_{\leq}), \alpha, (\Omega^\bullet_X(\log D), W, F), \beta),$$

where $X^* = X - D$ and $j : X^* \hookrightarrow X$. As was shown in Section 3.1, the Poincaré residue map gives a map of triples:

$$\begin{array}{c} (\mathrm{Gr}^{\tau_{\leq}}_m \mathcal{R}^\bullet j_* \mathbb{Q}_{X^*}, (\mathrm{Gr}^W_m \Omega^\bullet_X(\log D), F), \mathrm{Gr}_m(\beta)) \\ \big\downarrow \mathrm{Res}_m \\ ((a_m)_* \mathbb{Q}_{D^{[m]}}[-m](-m), ((a_m)_* \Omega^\bullet_{D^{[m]}}[-m], F[-m]), \mathbb{C}_{D^{[m]}}[0] \xrightarrow{\sim} \Omega^\bullet_{D^{[m]}}) \end{array}$$

constituted by maps that are isomorphisms in the derived category. By Deligne's results, the collection

$$(\mathbb{Q}_{D^{[m]}}[0], (\Omega^\bullet_{D^{[m]}}[0], F), \mathbb{C}_{D^{[m]}}[0] \xrightarrow{\sim} \Omega^\bullet_{D^{[m]}})$$

is a $\mathbb{Q}$-Hodge complex of sheaves of weight 0. (Recall that $D^{[m]}$ is compact Kähler.) This implies that $\mathrm{Gr}^W_m \mathcal{K}^\bullet_{DR}(X \log D)$ is a $\mathbb{Q}$-Hodge complex of sheaves of weight $-m + 2m = m$.

2.4.3. *The fundamental theorem of Deligne.*

Theorem 2.6 (Deligne, Fundamental Theorem)**.** Let $(\mathcal{K}^\bullet_{\mathbb{Z}}, (\mathcal{K}^\bullet_{\mathbb{Q}}, W), \alpha, (\mathcal{K}^\bullet_{\mathbb{C}}, W, F), \beta)$ be a mixed $\mathbb{Z}$-Hodge complex of sheaves on a topological space X. Set

$$K^\bullet_{\mathbb{Z}} := \mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Z}}), \quad K^\bullet_{\mathbb{Q}} := \mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), \quad K^\bullet_{\mathbb{C}} := \mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}),$$

and denote again by F, W the induced filtrations and by α, β the induced comparison morphisms. Then we have:

(1) The triple $(K^\bullet_{\mathbb{Z}}, (K^\bullet_{\mathbb{Q}}, W), \alpha, (K^\bullet_{\mathbb{C}}, W, F), \beta)$ is a mixed $\mathbb{Z}$-Hodge complex.
(2) The filtrations $\operatorname{Dec} W$, defined as $W[k]$ on the cohomology in degree k, and F induce a mixed Hodge structure on $\mathbb{H}^k(X, K^\bullet_{\mathbb{Z}})$. In fact we can say:
   (a) The filtration $W[k]$ on $\mathbb{H}^k(X, K^\bullet_{\mathbb{Q}})$ and the filtration induced by F on $\mathbb{H}^k(X, K^\bullet_{\mathbb{C}})$ define a mixed Hodge structure on $\mathbb{H}^k(X, K^\bullet_{\mathbb{Z}})$.
   (b) The first differential of the spectral sequence associated to $(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$ is strictly compatible with the filtration induced by F on $E_1(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$.
   (c) The spectral sequence $E_r(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{Q}}), W)$ degenerates at E_2.

(d) The spectral sequence ${}_F E_r$ associated to $(\mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}), F)$,
$$ {}_F E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}^p_F \mathcal{K}^\bullet_{\mathbb{C}}) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet_{\mathbb{C}}), $$
degenerates at ${}_F E_1$. Equivalently,
$$ \mathbb{H}^k(X, F^p\mathcal{K}^\bullet_{\mathbb{C}}) \longrightarrow \mathbb{H}^k(X, \mathcal{K}^\bullet_{\mathbb{C}}) $$
is always injective, and $\mathrm{Gr}^p_F \mathbb{H}^k(X, \mathcal{K}^\bullet_{\mathbb{C}}) \cong \mathbb{H}^k(X, \mathrm{Gr}^p_F \mathcal{K}^\bullet_{\mathbb{C}})$.

(e) The spectral sequence $E_r(\mathrm{Gr}^p_F \mathcal{R}\Gamma(X, \mathcal{K}^\bullet_{\mathbb{C}}), W)$ degenerates at E_2.

(3) Any morphism of mixed Hodge complexes of sheaves on X induces a morphism of mixed Hodge structures on the hypercohomology groups.

If we apply the theorem to $\mathcal{K}^\bullet_{DR}(X \log D)$, then we get the mixed Hodge structures on each $H^k(U)$.

2.4.4. *A useful lemma.*

Lemma 2.7 ([PS08] Lemma A.42)**.** Let $(K^\bullet, F)$ be a differential complex equipped with a biregular filtration. Then the following are equivalent:

(1) the differential d_K of $K^\bullet$ is strictly compatible with F, i.e., $\forall p, n$,
$$ \mathrm{d}_K(F^p(K^n)) = \mathrm{Im}(\mathrm{d}_K) \cap F^p(K^{n+1}). $$

(2) $\forall p, n$, the sequence
$$ 0 \to H^n(F^p(K)) \to H^n(K) \to H^n(K/F^pK) \to 0 $$
is exact.

(3) the spectral sequence $E_r(K^\bullet, F)$ degenerates at E_1.

3. Mixed Hodge Theory for Unitary Local Systems

Let X be a compact Kahler manifold with simple normal crossing divisor D, $U = X - D$, V a unitary local system on U. That is, V is a complex local system on U endowed with a Hermitian inner product $V \otimes \bar{V} \to \mathbb{C}$. Denote D_m = union of m-fold intersections of irreducible components of D, $D^{[m]}$ = normalization of D_m. Denote $a_m : D^{[m]} \to X$, and let $D(m) = a_m^{-1}(D_{m+1})$. We want to construct a mixed Hodge structure on the cohomologies $H^k(U, V)$. We will follow [Tim87].

First we want to replace $\Omega^\bullet_X(\log D)$ in the previous case by "$\Omega^\bullet_X(\log D) \otimes_{\mathbb{C}} V$", but since V is only a local system on U, this needs some careful construction. Luckily, this was explained in [Del70].

3.1. Canonical Extensions of Local Systems.

The goal of this subsection is to construct an extension of V to a holomorphic vector bundle with connection $(\mathcal{M}, \nabla)$ on X, with at most logarithmic poles along D. This extension we are going to construct is called the *canonical extension* of V.

We first consider a local model, that is, assume that X is a polydisc Δ^{n+m} with coordinates $(z_1, \ldots, z_{n+m})$ and that
$$ D = \{z_1 \cdots z_n = 0\}, \qquad X^* = (\Delta^*)^n \times \Delta^m. $$
The fundamental group $\pi_1(X^*) \cong \mathbb{Z}^n$ is generated by loops γ_i around $\{z_i = 0\}$. Let T_i be the monodromy transformation of V along γ_i. The T_i commute.

3.1.1. *Canonical extension for a unipotent local system.* Assume first that all T_i are *unipotent.* Then we can define
$$ U_i = -\frac{1}{2\pi i} \log T_i = \frac{1}{2\pi i} \sum_{k \geq 1} \frac{1}{k}(I - T_i)^k. $$
Set $\mathcal{M}$ to be the trivial vector bundle $\mathbb{C}^r \times X$ (where $r = \mathrm{rk}\, V$) with the connection
$$ \nabla = d + \sum_{i=1}^n U_i \frac{dz_i}{z_i}. $$
This connection is integrable because the U_i commute. Its horizontal sections are
$$ \exp\left(-\sum_{i=1}^n (\log z_i) U_i \right) \cdot v_0, $$
which are single-valued on the universal cover and, due to nilpotence, are polynomials in the $\log z_i$. In particular they grow at most like $(\log \|x\|)^k$ near D. Hence $\mathcal{M}$ extends V (the restriction to X^* is isomorphic to the original bundle) and has a logarithmic pole along D.

3.1.2. *General case.* Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a tuple of complex numbers, regarded as a representation $\lambda : \pi_1(U) \to \mathbb{C}^\times$. Let U_λ be the rank 1 local system on X^* with monodromy λ_i around γ_i. Its canonical extension (which is just a line bundle) can be chosen as follows: pick a determination of the logarithm, say $\log_\tau(x) = 2\pi i \tau(\frac{1}{2\pi i}\log(x))$ with τ a section of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ (we may choose τ so that $0 \leq \Re(\tau) < 1$). Set $\alpha_i = \frac{1}{2\pi i}\log_\tau \lambda_i$. Then the connection

$$d - \sum_i \alpha_i \frac{dz_i}{z_i}$$

defines a line bundle $\mathcal{L}_\lambda$ on X with logarithmic poles. This is the canonical extension of U_λ.

Now any local system V admits a unique decomposition into generalized eigenspaces of the commuting monodromies:

$$V \cong \bigoplus_\lambda U_\lambda \otimes V_\lambda,$$

where V_λ is a unipotent local system (all eigenvalues 1). Applying the unipotent construction to each V_λ, we obtain vector bundles with flat connections $(\mathcal{M}_\lambda, \nabla_\lambda)$ extending V_λ. The canonical extension of V is defined to be

$$\mathcal{M} = \bigoplus_\lambda \mathcal{L}_\lambda \otimes \mathcal{M}_\lambda.$$

The extension is unique in the following sense: if two extensions $\mathcal{M}_1$ and $\mathcal{M}_2$ of V to X satisfy that the horizontal sections (and their duals) have at most logarithmic growth (i.e. $\mathcal{O}((\log\|x\|)^k)$), then the identity map on X^* extends to an isomorphism $\mathcal{M}_1 \cong \mathcal{M}_2$. This is proved by comparing the identity map as a section of $\mathrm{Hom}(\mathcal{M}_1, \mathcal{M}_2)$: the growth condition forces it to be holomorphic across D. For details, see [Del70] Section II.5. Now since the construction is local (on coordinate charts where D is a union of coordinate hyperplanes), the uniqueness allows us to glue the local extensions to a global one on X. Therefore the canonical extension exists globally.

With the canonical extensions, one can define

Definition (Hodge filtration on $\Omega^\bullet_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{M}$)**.** $F^p\Omega^\bullet_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{M}$ is given by

$$0 \to \cdots \to 0 \to \Omega^p_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{M} \to \cdots \to \Omega^n_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{M} \to 0 \to \cdots.$$

But the definition of weight filtrations needs more preliminary constructions.

3.2. **Poincare Residues.** Define

$$V_m = j_*V|_{D_m \setminus D_{m+1}}.$$

Proposition 3.1. (i) V_m is a unitary local system on $D_m \setminus D_{m+1}$.

(ii) There exists a unique sub-vector bundle $\mathcal{M}_m$ of $a_m^*\mathcal{M}$ and a unique holomorphic integrable connection ∇_m on $\mathcal{M}_m$ with logarithmic poles along $D(m)$ such that

$$\ker\nabla_m|_{D^{[m]}\setminus D(m)} = a_m^{-1}V_m.$$

(iii) On $D^{[m]}$, $(\mathcal{M}_m, \nabla_m)$ is the canonical extension of $a_m^{-1}V_m$.

(iv) There exists a unique sub-vector bundle $\mathcal{M}_m^*$ of $a_m^*\mathcal{M}$ with

$$a_m^*\mathcal{M} = \mathcal{M}_m \oplus \mathcal{M}_m^*$$

such that for any $x \in D^{[m]}\setminus D(m)$,

$$\mathcal{M}^*_{m,x} = a_m^*(\{\sigma \in \mathcal{M}_{a_m(x)} : (\sigma, \tau) = 0, \forall \tau \in (j_*V)_{a_m(x)}\}).$$

Proof. The Hermitian form on j_*V is inherited from the one on V. So for (1) it suffices to show V_m is a local system on $D_m \setminus D_{m+1}$. The uniqueness assertions in (2), (4) are clear, so it suffices to prove the proposition locally, that is, for $X = \Delta^n, D = \pi_1^{-1}(0) \cup \cdots \cup \pi_s^{-1}(0)$. Denote $D_j = \pi_j^{-1}(0)$. Denote by T_i the monodromy operator along the loop that encircles D_i. Since $\pi_1(X\setminus D)$ is abelian, and V is unitary, we can diagonalize all T_i simultaneously. So we have

$$V|_{X\setminus D} = \bigoplus_{i=1}^r V^i|_{X\setminus D},$$

with all V^i being rank 1 local systems. Suppose T_j acts on V^i by $\gamma_{ij} \in \mathbb{C}^\times$, then $|\gamma_{ij}| = 1$. If some $\gamma_{ij} = 1$, then T_j acts by id on V^i, and we can extend the local system V^i over D_j.

By construction of the canonical extension, we have

$$(\mathcal{M},\nabla)=\bigoplus_{i=1}^{r}(\mathcal{M}^i,\nabla^i),$$

where $(\mathcal{M}^i,\nabla^i)$ is the canonical extension of V^i.

(i) Let us consider the local behaviour of j_*V^i near $y\in D_m\backslash D_{m+1}$. Suppose $y\in D_{j_1}\cap\cdots\cap D_{j_m}$ Take a small neighbourhood Δ_y of y, so that

$$\Delta_y\backslash D\cong(\Delta^*)^m\times\Delta^{n-m}.$$

Then

$$j_*V^i(\Delta_y)=V^i(\Delta_y\backslash D)=\begin{cases}\mathbb{C}, & T_{j_1},\dots,T_{j_m}\text{ act by identity, i.e. }\gamma_{i,j_1}=\cdots=\gamma_{i,j_m}=0\\ 0, & \text{else.}\end{cases}\tag{1}$$

So

$$V_m|_{\Delta_y}=j_*V|_{(D_m\backslash D_{m+1})\cap\Delta_y}=\bigoplus_{i=1}^{r}j_*V^i|_{(D_m\backslash D_{m+1})\cap\Delta_y}$$

is a constant sheaf, and V_m is a local system.

(ii) By (1) we can write

$$V_m=\bigoplus_{\substack{1\le i\le r\\ \gamma_{i,j_1}=\cdots=\gamma_{i,j_m}=1}}j_*V^i|_{(D_m\backslash D_{m+1})}.$$

Thus

$$(a_m^{-1}V_m)|_{D_J\backslash D(J)}=a_m^{-1}(\bigoplus_{\substack{1\le i\le r\\ \gamma_{i,j_1}=\cdots=\gamma_{i,j_m}=1}}j_*V^i|_{(D_J\backslash D(J))}).$$

So we define

$$\mathcal{M}_m|_{D_J}=a_m^*(\bigoplus_{\substack{1\le i\le r\\ \gamma_{i,j_1}=\cdots=\gamma_{i,j_m}=1}}\mathcal{M}^i|_{D_J}).$$

The connection ∇^i on $\mathcal{M}^i$ has logarithmic singularities along D_j (with $\gamma_{ij}\neq 1$). So the connetion ∇_m on $\mathcal{M}_m$ induced by the ∇^i's has logarithmic singularities along $D(J)$. By construction,

$$\ker\nabla_m|_{D_J\backslash D(J)}=(a_m^{-1}V_m)|_{D_J\backslash D(J)}.$$

(iii) is clear from (ii).

(iv) Suppose $x\in D_J\subset D^{[m]}$. Then we have

$$\{\sigma\in\mathcal{M}_{a_m(x)}:(\sigma,\tau)=0,\forall\tau\in(j_*V)_{a_m(x)}\}=\bigoplus_{\substack{1\le i\le r\\ \gamma_{i,j_l}\neq 1\text{ for some }l}}\mathcal{M}^i_{a_m(x)}.$$

So define

$$\mathcal{M}^*_m|_{D_J}=a_m^*(\bigoplus_{\substack{1\le i\le r\\ \gamma_{i,j_l}\neq 1\text{ for some }l}}\mathcal{M}^i|_{D_J}).$$

□

Now we define $\mathrm{Res}_m(\mathcal{M}):\Omega^q_X(\log D)\otimes_{\mathcal{O}_X}\mathcal{M}\to(a_m)_*(\Omega^{q-m}_{D^{[m]}}(\log D(m))\otimes_{\mathcal{O}_{D^{[m]}}}\mathcal{M}_m)$ as the composition

$$\begin{aligned}\Omega^q_X(\log D)\otimes_{\mathcal{O}_X}\mathcal{M}&\to(a_m)_*\Omega^{q-m}_{D^{[m]}}(\log D(m))\otimes_{\mathcal{O}_X}\mathcal{M}\\&=(a_m)_*(\Omega^{q-m}_{D^{[m]}}(\log D(m))\otimes_{\mathcal{O}_{D^{[m]}}}a_m^*\mathcal{M})\\&\to(a_m)_*(\Omega^{q-m}_{D^{[m]}}(\log D(m))\otimes_{\mathcal{O}_{D^{[m]}}}\mathcal{M}_m).\end{aligned}$$

Lemma 3.2. $\mathrm{Res}_m(\mathcal{M})\circ\nabla=\nabla_m\circ\mathrm{Res}_m(\mathcal{M})$.

Definition (Weight filtration on $\Omega^\bullet_X(\log D)\otimes_{\mathcal{O}_X}\mathcal{M}$)**.** For $m<0$, set $W_m(\Omega^\bullet_X(\log D)\otimes_{\mathcal{O}_X}\mathcal{M})=0$. For $m\ge 0$, define

$$W_m(\Omega^\bullet_X(\log D)\otimes_{\mathcal{O}_X}\mathcal{M}):=\ker\mathrm{Res}_{m+1}(\mathcal{M}).$$

From the preceding, we clearly have the following local description of $W_m(\Omega_X^q(\log D) \otimes \mathcal{M})$ for a local system of rank 1: If $\Delta \subseteq X$ is a polycylinder with coordinates $z_1, \ldots, z_n$ such that $D \cap \Delta$ is given by $z_1 \cdots z_s = 0$, and if T_j is the monodromy transformation of $V|_{\Delta \setminus D}$ around $\{z_j = 0\}$, then $W_m(\Omega_X^q(\log D) \otimes \mathcal{M})|_\Delta$ is generated over $\mathcal{O}_\Delta$ by $\Omega_X^q \otimes \mathcal{M}|_\Delta$ and by all terms of the form

$$\frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \wedge dz_{j_{k+1}} \wedge \cdots \wedge dz_{j_q} \otimes \mu$$

where $\mu \in \Gamma(\Delta, \mathcal{M})$, $k \leq q$, $1 \leq j_1 < \cdots < j_k \leq s < j_{k+1} < \cdots < j_q \leq n$, and for at most m indices κ with $1 \leq \kappa \leq k$ we have $T_{j_\kappa} = 1$.

3.3. **Quasi-isomorphisms of Complexes.**

Proposition 3.3. $\mathrm{Res}_m(\mathcal{M})$ induces an isomorphism

$$\mathrm{Gr}_m^W(\Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_X} \mathcal{M}) \cong (a_m)_*(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m))[-m].$$

Proposition 3.4. The inclusion map

$$(\Omega_X^\bullet(\log D), \tau_\leq) \to (\Omega_X^\bullet(\log D), W_\bullet)$$

is a filtered quasi-isomorphism.

Proposition 3.5. $\Omega_X^\bullet(\log D) \otimes \mathcal{M} \simeq Rj_*V$.

Proof. V admits a resolution $\Omega_U^\bullet \otimes_{\mathbb{C}} V \cong \Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{M}|_U$. So $Rj_*V \simeq Rj_*(\Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{M}|_U) \simeq j_*(\Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{M}|_U)$. By [Del70] Chapter II, Corollary 3.14, we have a quasi-isomorphism $\Omega_X^\bullet(\log D) \otimes \mathcal{M} \to j_*(\Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{M}|_U)$. ☐

Proposition 3.6. For any m, $\mathrm{Gr}_m^W(\Omega_X^\bullet(\log D) \otimes \mathcal{M})$ and $R^m j_*V[-m]$ are quasi-isomorphic. Moreover, $R^m j_* V \cong (j_m)_* V_m$, where $j_m : D_m \backslash D_{m+1} \to X$.

Proof. Here we only prove the first statement. We have $\mathrm{Gr}_m^W(\Omega_X^\bullet(\log D) \otimes \mathcal{M}) \simeq \mathrm{Gr}_m^{\tau_\leq}(\Omega_X^\bullet(\log D) \otimes \mathcal{M}) \simeq h^m((\Omega_X^\bullet(\log D) \otimes \mathcal{M}))[-m] \cong R^m j_* V[-m]$. ☐

In particular, taking $m = 0$ we see $\tilde{\Omega}_X^\bullet(\mathcal{M}) := W_0\Omega_X^\bullet(\log D)$ and j_*V are quasi-isomorphic.

By the results above, the following are quasi-isomorphic:

$$\mathrm{Gr}_m^W(\Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_X} \mathcal{M}), (a_m)_*(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m))[-m], R^m j_* V[-m], (j_m)_* V_m[-m].$$

If we consider A-structures, with $A \subset \mathbb{C}$ noetherian such that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field, and assume V is A-unitary, that is, V is induced from an A-local system V_A with symmetric positive-definite bilinear form $(,)_A$ by $V \cong V_A \otimes_A \mathbb{C}$, with Hermitian inner product given by $(v \otimes c, w \otimes c') = c\overline{c'}(v, w)_A$. Then V_m are all A-unitary, and

Lemma 3.7. The isomorphism $R^m j_* V \to (j_m)_* V_m$ maps $R^m j_* V_{A \otimes \mathbb{Q}}$ to

$$(j_m)_* V_{m, A\otimes\mathbb{Q}}(-m) := (2\pi i)^{-m} (j_m)_* V_{m, A\otimes\mathbb{Q}} \subset (j_m)_* V_m.$$

3.4. **Pure Hodge Structure on $H^k(X, j_*V)$.** Denote by $F^\bullet$ the Hodge filtration on $H^k(X, j_*V)$, induced by the Hodge filtration on $\tilde{\Omega}_X^\bullet(\mathcal{M})$. Let $\bar{F}^\bullet$ be the lift under the conjugation map $H^k(X, j_*V) \to H^k(X, j_*V^\vee)$ of the Hodge filtration on $H^k(X, j_*V^\vee)$, and let

$$H^{p,q}(X, j_*V) := (F^p \cap \bar{F}^p) H^{p+q}(X, j_*V).$$

Theorem 3.8. Let X be a compact Kähler manifold, $D \subseteq X$ a simple normal crossing divisor, $U := X \setminus D$, $j : U \hookrightarrow X$ the inclusion map, V a unitary local system on U, $\mathcal{M}$ the canonical extension of V.

(i) The spectral sequence

$$E_1^{p,q} = H^q(X, \tilde{\Omega}_X^p(\mathcal{M})) \implies H^{p+q}(X, j_*V)$$

degenerates at E_1.

(ii) The maps $H^{p,q}(X, j_*V) \to H^q(X, \tilde{\Omega}_X^p(\mathcal{M}))$ are isomorphisms, and

$$H^k(X, j_*V) = \bigoplus_{p+q=k} H^{p,q}(X, j_*V).$$

(iii) The conjugation map induces conjugate linear isomorphisms
$$H^q(X, \tilde{\Omega}^p_X(\mathcal{M})) \cong H^p(X, \tilde{\Omega}^q_X(\mathcal{N}))$$
where $\mathcal{N}$ is the canonical extension of $V^\vee$.

3.5. **Mixed Hodge Structure on $H^k(U,V)$.** Now we can finally tackle our main problem for this section. Let us first define the Hodge and weight filtrations on $H^k(U,V)$:

Definition.
$$F^p H^k(U,V) := \operatorname{im}\left(\mathbb{H}^k\big(X, F^p(\Omega^\bullet_X(\log D)\otimes\mathcal{M})\big) \longrightarrow \mathbb{H}^k\big(X, \Omega^\bullet_X(\log D)\otimes\mathcal{M}\big) = H^k(U,V)\right);$$
For the weight filtration, one uses a shift by k:
$$W_{m+k} H^k(U,V) := \operatorname{im}\left(\mathbb{H}^k\big(X, W_m(\Omega^\bullet_X(\log D)\otimes\mathcal{M})\big) \longrightarrow \mathbb{H}^k\big(X, \Omega^\bullet_X(\log D)\otimes\mathcal{M}\big) = H^k(U,V)\right).$$

The following lemma is an immediate consequence of the definitions and the description of the weight filtration on $\Omega^\bullet_X(\log D)\otimes\mathcal{M}$:

Lemma 3.9. (i) $W_i H^k(U,V) = 0$ for $i < k$.
(ii) $W_k H^k(U,V) = \operatorname{im}\big(H^k(X, j_*V) \to H^k(U,V)\big)$.
(iii) Let m_0 be the largest $m \in \mathbb{N}$ such that there exist $x \in X$ and m local components of D through x such that the monodromy of V around these components has an eigenvalue 1. Then
$$W_{k+m_0} H^k(U,V) = H^k(U,V).$$

We now assume that V is A-unitary with $A \subseteq \mathbb{R}$. Our aim is to prove:

Theorem 3.10. Let X be a compact Kähler manifold, $D \subseteq X$ a simple normal crossing divisor, $U := X \setminus D$. Let V be an A-unitary local system on U, with a subring $A \subseteq \mathbb{R}$. Then the Hodge and weight filtrations on $H^k(U,V)$ give a mixed Hodge structure on $H^k(U,V_A)$.

By Theorem 2.6, it suffices to show

Proposition 3.11.
$$\big(Rj_*V_A,\ (Rj_*V_{A\otimes\mathbb{Q}}, \tau),\ (\Omega^\bullet_X(\log D)\otimes\mathcal{M}, F^\bullet, W_\bullet)\big)$$
is a mixed A-Hodge complex of sheaves.

Proof. We have $(Rj_*V_{A\otimes\mathbb{Q}}, \tau_\leq)\otimes\mathbb{C} \cong (Rj_*V_\mathbb{C}, \tau_\leq) \simeq (\Omega^\bullet_X(\log D)\otimes\mathcal{M}, \tau_\leq) \simeq (\Omega^\bullet_X(\log D)\otimes\mathcal{M}, W_\bullet)$. So it remains to show that
$$\big(\operatorname{Gr}^{\tau_\leq}_m(Rj_*V_{A\otimes\mathbb{Q}}),\ (\operatorname{Gr}^W_m(\Omega^\bullet_X(\log D)\otimes\mathcal{M}), F^\bullet)\big)$$
is an $A\otimes\mathbb{Q}$-Hodge complex of weight m, i.e. the spectral sequence associated to the filtration of the second complex degenerates at E_1, and the induced filtration on
$$\mathbb{H}^k\big(X, \operatorname{Gr}^W_m(\Omega^\bullet_X(\log D)\otimes\mathcal{M})\big) \cong \mathbb{H}^k\big(X, \operatorname{Gr}^{\tau_\leq}_m(Rj_*V_{A\otimes\mathbb{Q}})\big)\otimes\mathbb{C}$$
defines a pure $A\otimes\mathbb{Q}$-Hodge structure of weight $m+k$ on $\mathbb{H}^k\big(\operatorname{Gr}^W_m(Rj_*V_{A\otimes\mathbb{Q}})\big)$.

We have
$$\operatorname{Gr}^{\tau_\leq}_m(Rj_*V_{A\otimes\mathbb{Q}}) \simeq R^m j_* V_{A\otimes\mathbb{Q}}[-m] \simeq (j_m)_*V_{m,A\otimes\mathbb{Q}}(-m)[-m].$$

For the second complex, we use the isomorphism in Proposition 3.3:
$$\operatorname{Gr}^W_m(\Omega^\bullet_X(\log D)\otimes_{\mathcal{O}_X}\mathcal{M}) \cong (a_m)_*(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m))[-m].$$
$$\big(\operatorname{Gr}^W_m(\Omega^\bullet_X(\log D)\otimes\mathcal{M}), F^\bullet\big) \xrightarrow{\operatorname{Res}_m(\mathcal{M})} \big((a_m)_*(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m)), \tilde{F}^\bullet\big)[-m],$$
where $\tilde{F}^\bullet(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m))$ is given by
$$\tilde{F}^p(\tilde{\Omega}^q_{D^{[m]}}(\mathcal{M}_m)) = \begin{cases} 0, & q+m < p \\ \tilde{\Omega}^q_{D^{[m]}}(\mathcal{M}_m), & q+m \geq p. \end{cases}$$

Hence we have to show that
$$\Big((j_m)_*V_{m,A\otimes\mathbb{Q}}(-m)[-m], \big((a_m)_*(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m)), \tilde{F}^\bullet\big)[-m]\Big)$$

is an $A \otimes \mathbb{Q}$-Hodge complex of weight m. This follows if

$$\left((j_m)_* V_{m,A\otimes\mathbb{Q}}, \big((a_m)_*(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m)), F^\bullet\big) \right)$$

(with $F^\bullet$ the usual Hodge filtration) is an $A \otimes \mathbb{Q}$-Hodge complex of weight 0. Write $(j_m)_* V_{m,A\otimes\mathbb{Q}} = (a_m)_*(\tilde{j}_m)_*(a_m)^{-1} V_{m,A\otimes\mathbb{Q}}$, where $\tilde{j}_m : D^{[m]} \backslash D(m) \to D^{[m]}$. Then it suffices to show

$$\left((\tilde{j}_m)_*(a_m)^{-1} V_{m,A\otimes\mathbb{Q}}, \big(\tilde{\Omega}^\bullet_{D^{[m]}}(\mathcal{M}_m), F^\bullet\big) \right)$$

is an $A \otimes \mathbb{Q}$-Hodge complex of weight 0. But this just follows from Theorem 3.8, applied to $D^{[m]}$ and the local system $(a_m)^{-1} V_{m,A\otimes\mathbb{Q}}$. □

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