Outline :

§0 Motivation: from singularity theory

- Milnor fibration & monodromy action
- The case of isolated (quasi-homogeneous) singularities
- The case of homogeneous singularities $\rightsquigarrow$ hypersurface complement

§1 Combinatorics of hyperplane arrangements

- Intersection lattice & Poincaré polynomial
- Known results on the combinatorial determinacy of topological invariants

§2. Combinatorial description of $H^*(M(\mathcal{A}), \mathbb{Z})$

- Orlik - Solomon algebra $OS^*(\mathcal{A})$
- Proof of sketch

§3 Open Problems : Is $H^*(M(\mathcal{A}), \mathbb{L})$ combinatorially determined?

- Significance . Cohomology of Milnor fiber
- Known results & Main conjecture

§0 Motivation: from singularity theory

$f : (\mathbb{C}^{n+1}_x, 0) \longrightarrow (\mathbb{C}_t, 0)$ a germ of holom. func. w/ critical value 0

Choose $0 < \delta \ll \varepsilon \ll 1$, then

$$\begin{array}{ccc} X^* = X \setminus f^{-1}(0) & \hookrightarrow & X = \{|x|<\varepsilon\} \cap f^{-1}(D) \\ \downarrow f & & \downarrow f \\ D^* = D \setminus \{0\} & \hookrightarrow & D = \{|t|<\delta\} \end{array}$$

Milnor 1968

$f : X^* \longrightarrow D^*$ is a smooth fibration

$\Updownarrow$ up to homotopy, determined by

fiber F & monodromy ρ (topological invariants of f)

$\Updownarrow$ at the cohomology level

$H^p(F, \mathbb{C})$ & $T = (\rho^*)^{-1} \in \mathrm{Aut}(H^p(F, \mathbb{C}))$

$\Updownarrow$

local system $\underline{H}^p = R^p f_* \underline{\mathbb{C}}_{X^*}$ on D^* (Cohomological Milnor fibration)

$\Updownarrow$ Riemann-Hilbert correspondence

vector bundle $\mathcal{H}^p = \underline{H}^p \otimes \mathcal{O}_{D^*}$ + flat connection ∇ (Topological Gauss-Manin connection)

Example: If 0 is an isolated singularity of f.

then the singular fiber X_0 is homeomorphic to a cone over 0,

and F is homotopic to a bouquet of some spheres.

$\mu(f) \triangleq \#$ spheres $= \dim H^n(F, \mathbb{C}) = \mathrm{rank}_{\mathbb{C}} \mathcal{H}^n = \dim_{\mathbb{C}} \mathcal{O}_{X,0}/J_f$

Note that $\nabla : \mathcal{H}^n \longrightarrow \Omega^1_{D^*} \otimes \mathcal{H}^n$ can be extended to

$$f_*\Omega^n_{X/D} / d(f_*\Omega^{n-1}_{X/D}) = {}'\mathcal{H} \longrightarrow \Omega^1_D \otimes {}''\mathcal{H} = f_*\Omega^{n+1}_X / df \wedge d(f_*\Omega^n_X) \text{ (Brieskorn lattice)}$$

$(\Omega^p_{X/D} = \Omega^p_X / df \wedge \Omega^{p-1}_X)$

$$[\eta] \longmapsto dt \otimes [d\eta] \qquad \text{(Algebraic Gauss-Manin connection)}$$

where ${}'\mathcal{H} \xrightarrow{df \wedge} {}''\mathcal{H}$ are both coherent, even locally free of rank $\mu(f)$ over D.

Nontrivial!

Due to 0 is an isolated singularity.

In particular, when f is homogeneous of degree d:

Let $\xi = \frac{1}{d}\sum_{i=0}^{n}(-1)^i x_i\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. Then $f dx = df \wedge \xi$

So locally at $0 \xRightarrow{\text{Nakayama}} (df \wedge {}'\mathcal{H})_0 = t \cdot {}''\mathcal{H}_0$, and any $\mathbb{C}$-basis of ${}''\mathcal{H}_0 / df \wedge {}'\mathcal{H}_0 \simeq \mathcal{O}_{X,0}/J_f$ is an $\mathbb{C}\{t\}$-basis of ${}''\mathcal{H}_0$

Choose the monomial basis A of $\mathcal{O}_{X,0}/J_f$.

Then ${}''\mathcal{H}_0 = \bigoplus_{x^m \in A} \mathbb{C}\{t\}\, x^m dx$. ${}'\mathcal{H}_0 = \bigoplus_{x^m \in A} \mathbb{C}\{t\} \cdot x^m \xi$, and $df \wedge x^m \xi = f \cdot x^m dx$.

By definition, ∇ is given by

$$\nabla [x^m \xi] = dt \otimes \Big(\sum_{i=0}^{n} \frac{m_i+1}{d} \Big)[x^m dx]$$

In particular, T is semisimple with eigenvalues $\{ \exp(-2\pi i \sum_{i=0}^{n} \frac{m_i+1}{d}) \mid x^m \in A \}$

Non-isolated cases are much more complicated. We still focus on homogeneous singularities. In this case, we have the affine Milnor fibration

$$\begin{array}{ccc} \mathbb{C}^{n+1} \setminus f^{-1}(0) & \hookrightarrow & \mathbb{C}^{n+1} \\ \downarrow f & & \downarrow f \\ \mathbb{C} \setminus \{0\} & \hookrightarrow & \mathbb{C} \end{array}$$

which is fiber diffeomorphic to the Milnor fibration.

So the Milnor fiber F is diffeomorphic to $f^{-1}(1)$, which admits a cyclic cover of degree $d = \deg f$ to the hypersurface complement $\mathbb{P}^n \setminus \{f = 0\} \overset{\Delta}{=} M(\mathcal{A})$.

Simplest case : hyperplane arrangements ($f = \prod L_i^{d_i}$. L_i linear) rich combinatorial structure

$\updownarrow$

$\mathcal{A} = \{ H_i = \{ L_i = 0 \} \} + d$

§ 1 Combinatorics of hyperplane arrangements

The combinatorial data of $\mathcal{A}$ consists of :

- a finite set $L(\mathcal{A}) = \{$ intersections of hyperplanes in $\mathcal{A} \}$
- a partial order reverse inclusion : $X \leq Y \Leftrightarrow Y \subseteq X$

admitting least upper bounds (join) $X \vee Y = X \cap Y$

& greatest lower bounds (meet) $X \wedge Y = \bigcap_{\substack{H \in \mathcal{A} \\ X, Y \subseteq H}} H$

for each pair (X, Y) in th set, and

admitting a rank function r taking value in $\mathbb{N}$ s.t $r(Z) = \operatorname{codim} Z$

(1) $X \prec Y \Rightarrow r(X) < r(Y)$; (2) $r(X) < r(Y) - 1 \Rightarrow \exists Z.\ X < Z \prec Y$

Such pair forms a ranked lattice, called the intersection lattice of $\mathcal{A}$

Prop : $rk(X) + rk(Y) \geq rk(X \vee Y) + rk(X \wedge Y)$, $\forall X, Y \in L(\mathcal{A})$

For a hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$, its defining equations form a point configuration in $\mathbb{P}((\mathbb{C}^{n+1})^\vee)$ $\rightsquigarrow$ A matroid

This gives rise to :

- dependent : $S \subseteq \mathcal{A}$ s.t. $\underbrace{r(\bigcap_{H \in S} H)}_{= \text{rank of } S} = \operatorname{codim}(\bigcap_{H \in S} H) < \# S$
- independent : - - — — — — — — — — — - - $=$ . . .
- circuit : minimal dependent
- closed : $\mathcal{F} \subseteq \mathcal{A}$ s.t. for any circuit C, $|\mathcal{F} \cap C| \geq |C| - 1 \Rightarrow C \subseteq \mathcal{F}$

  so-called "flats". 1-1 correspondence with elements $(\bigcap_{H \in \mathcal{F}} H)$ in $L(\mathcal{A})$

  $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F} \vee \mathcal{G} =$ closure of $\mathcal{F} \cup \mathcal{G}$, $\mathcal{F} \wedge \mathcal{G} = \mathcal{F} \cap \mathcal{G}$

Any one of them determines $L(\mathcal{A})$, vice versa.

Möbius function: $\mu : L(\mathcal{A}) \to \mathbb{Z}$, $\mu(\mathbb{P}^n) = 0$, $\sum_{Y \le X} \mu(Y) = 0$. $\forall X > \mathbb{P}^n$

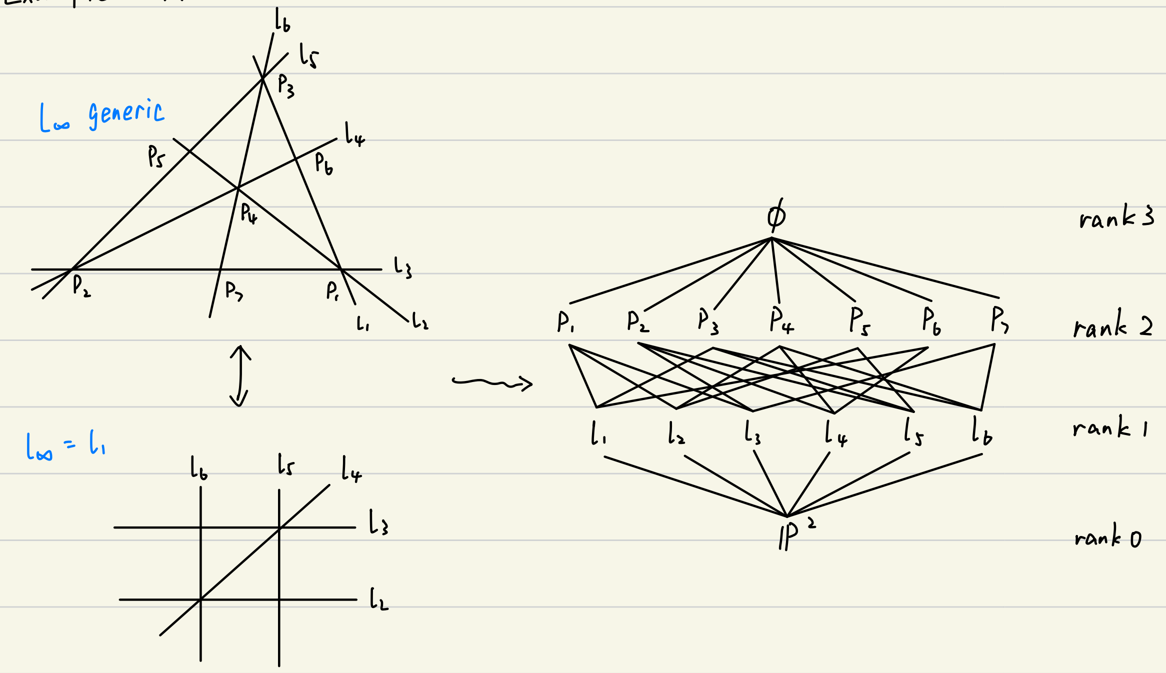
Characteristic polynomial $\chi_{\mathcal{A}}(t) = \sum_{Z \in L(\mathcal{A})} \mu(Z) t^{n+1-r(Z)}$

Poincaré polynomial $\pi_{\mathcal{A}}(t) = \sum_{Z \in L(\mathcal{A})} \mu(Z)(-t)^{r(Z)} = (-t)^{n+1} \chi(-\frac{1}{t})$

Reduced ones: $\overline{\chi_{\mathcal{A}}} = \frac{\chi_{\mathcal{A}}}{t-1}$. $\overline{\pi}_{\mathcal{A}} = \frac{\pi_{\mathcal{A}}}{t+1}$

- For real $\mathcal{A}$, $\overline{\pi}_{\mathcal{A}}(1)$. $\overline{\pi}_{\mathcal{A}}(-1)$ count "# of regions"

Example: A3



circuits: $\{1,2,3\}$, $\{3,4,5\}$, $\{1,5,6\}$, $\{2,4,6\}$

$\chi_{A_3}(t) = (t-1)(t-2)(t-3)$. $\pi_{A_3}(t) = (1+t)(1+2t)(1+3t)$

$\overline{\pi}(1) = 12 = $ # of conn. components; $\overline{\pi}(-1) = 2 = $ # "bounded region"

Q : Which topological invariants of $M(\mathcal{A}) = (\mathbb{P}^n \setminus \bigcup_{H \in \mathcal{A}} H)$ are determined by $L(\mathcal{A})$?

Nontrivial ! Since "moduli of $\mathcal{A}$ with certain $L(\mathcal{A})$" is not connected !

$G = \pi_1(M(\mathcal{A}))$ ✗ Rybnikov 1994

local (co)homology

$H(M(\mathcal{A}), \mathbb{L})$? ☆

$\mathbb{L}$: rank-1 $\mathbb{K}$-local system with monodromy α

variation of α

group structure of G

LCS quotients $gr_k G$ ✗

EBJ 2020

Characteristic varieties $V_*(G, \mathbb{K})$?

Arapura 1997

Resonance varieties $R_*(G, \mathbb{K})$ ✓

Chen groups $gr_k(G/G'')$?

AFRS 2024

Alexander matrices

LCS ranks $\phi_k(G)$ ✓

Need a formula

$\pi_1(M(\mathcal{A})) \otimes \mathbb{Q}$ ✓

$H(M(\mathcal{A}), \mathbb{Z})$ ✓

Orlik & Solomon 1980

Brieskorn 1973 : $M(\mathcal{A})$ is a formal space

Higher homotopy : $\pi_n(M(\mathcal{A}))$? $K(\pi, 1)$ property ? $\pi_n(M(\mathcal{A})) \otimes \mathbb{Q}$ ✓

Notion :

Given a group G, $\gamma_1(G) = G$, $\gamma_k(G) = [\gamma_{k-1}(G), G]$, $gr_k G = \gamma_k G / \gamma_{k+1}(G)$ (lower central series)

and $G \otimes \mathbb{Q} = \bigoplus_{k \geq 0} gr_k G$ equipped with commutators as the lie bracket.

§ 2. Combinatorial description of $H^*(M(\mathcal{A}), \mathbb{Z})$

We first state the conclusion: (Always assume that $\mathcal{A} \neq \emptyset$!)

Def. For $\mathcal{A} = \{H_1, \cdots, H_r\}$ in $\mathbb{P}^n$, denote by $E = E^*(\mathcal{A}) = \Lambda^*(\bigoplus_{i=1}^r \mathbb{Z} e_i)$

For any $S = \{H_{i_1}, \cdots, H_{i_k}\} \subset \mathcal{A}$ $(i_1 < \cdots < i_k)$, denote by $e_S = e_{i_1} \wedge \cdots \wedge e_{i_k} \in E^k$

Define:

graded derivation $\partial : E \to E$, $\partial e_S = \sum_{j=1}^k (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}$

Orlik-Solomon ideal $I(\mathcal{A})$ = the ideal generated by $\{\partial e_S \mid S \text{ dependent}\}$

Orlik-Solomon algebra $OS^*(\mathcal{A}) = E / I(\mathcal{A})$

$I(\mathcal{A}) \subset \ker \partial$ since $\partial^2 = 0$

reduced (projective) Orlik-Solomon algebra $\overline{OS}^*(\mathcal{A}) = \ker \partial / I(\mathcal{A})$

Thm (OS80): Fix a defining linear functional α_i for each $H_i \in \mathcal{A}$.

$\{\alpha_i\}$ defines a central hyperplane arrangement $c\mathcal{A}$ in $\mathbb{C}^{n+1}$ cone of $\mathcal{A}$

There exists a graded $\mathbb{Z}$-algebra isomorphism $OS^*(\mathcal{A}) \xrightarrow{\sim} H^*(M(c\mathcal{A}), \mathbb{Z})$

mapping e_i to $[\omega_i] = \left[\frac{1}{2\pi\sqrt{-1}} \frac{d\alpha_i}{\alpha_i}\right]$.

Rmk: 1 $\mathcal{A}$ is the projectivization $\overline{c\mathcal{A}}$ of it cone. with same combinatorial data

Since $\mathcal{A} \neq \emptyset$, we have $M(c\mathcal{A}) \simeq M(\mathcal{A}) \times \mathbb{C}^\times$.

In fact, we have the commutative diagram since $\omega_i(\theta) = 1$ for $\theta = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$

$$\begin{array}{ccc}
\overline{OS}^*(\mathcal{A}) & \xrightarrow{\sim} & H^*(M(\mathcal{A}), \mathbb{Z}) \\
\downarrow & & \downarrow \\
\overline{OS}^*(\mathcal{A}) \otimes (\mathbb{C} \oplus \mathbb{C}[-1]) = OS^*(\mathcal{A}) & \xrightarrow{\sim} & H^*(M(c\mathcal{A}), \mathbb{Z}) = H^*(M(\mathcal{A}), \mathbb{Z}) \otimes (\mathbb{C} \oplus \mathbb{C}[-1]) \\
e_i & \longmapsto & [\omega_i]
\end{array}$$

(So maybe a more reasonable notion is $A^*(\mathcal{A}) = \begin{cases} \overline{OS}^*(\mathcal{A}), & \text{when } \mathcal{A} \text{ is projective} \\ OS^*(\overline{\mathcal{A}}), & \text{when } \mathcal{A} \text{ is central} \end{cases}$)

2. $\forall S \subset \mathcal{A}$, if $\partial e_S \in I(\mathcal{A})$, then:

$$e_S = e_{i_1} \wedge \partial e_S \in I(\mathcal{A})$$

$$\partial(e_H e_S) = \partial e_H \cdot e_S - e_H \wedge \partial e_S = e_S - e_H \wedge \partial e_S \in I(\mathcal{A})$$

As a corollary, $I(\mathcal{A})$ is generated by $\{\partial e_C \mid C \text{ circuit}\}$ as an ideal

3. $[w_i]$ is given by the pull-back from $M(c\mathcal{A}) \hookrightarrow \mathbb{C}^{n+1} \setminus \{cH_i\} \simeq \mathbb{C}^n \times \mathbb{C}^\times$,

which is independent of the choice of α_i.

Denote by $\mathbb{R}$ the subalgebra of $H^*(M(c\mathcal{A}), \mathbb{Z})$ generated by $\{[w_i]\}$

Brieskorn proved in 1973 that $\mathbb{R} = H^*(M(c\mathcal{A}), \mathbb{Z})$ by induction on dimension.

Proof of sketch: In this part, $\mathcal{A}$ will be central.

Well-definedness follows trivially from the fact that:

$$w_S \triangleq \sum_{j=1}^{k} (-1)^{j-1} w_{i_1} \wedge \cdots \wedge \widehat{w_{i_j}} \wedge \cdots \wedge w_{i_k} = \frac{1}{\prod_{j=1}^{k} \alpha_{i_j}} \cdot \sum_{j=1}^{k} (-1)^{j-1} \alpha_{i_j} \cdot d\alpha_{i_1} \wedge \cdots \wedge \widehat{d\alpha_{i_j}} \wedge \cdots \wedge d\alpha_{i_k} = 0$$

∵ "local vanishing condition is all you need"

Now we prove the conclusion by induction via

<u>Deletion-Restriction</u>

$\forall$ flat $\mathcal{F} \subset \mathcal{A} \rightsquigarrow$ subspace $W = \bigcap_{H \in \mathcal{F}} H \subset V = \mathbb{C}^{n+1}$

$H \supseteq W,\ H \in \mathcal{A}$ $\Updownarrow$ $\qquad\qquad$ $H \not\supseteq W,\ H \in \mathcal{A}$ $\Updownarrow$

$\mathcal{A}_{\mathcal{F}} = \{H/W \mid H \in \mathcal{F}\}$ in V/W $\qquad\qquad$ $\mathcal{A}^{\mathcal{F}} = \{H \cap W \mid H \in \mathcal{A} \setminus \mathcal{F}\}$ in W

localization $\qquad\qquad$ restriction $\uparrow$ need this

Deletion-Restriction triple: $(\mathcal{A} \setminus H, \mathcal{A}, \mathcal{A}^H)$ for $H \in \mathcal{A}$

Fix $H = H_1$:

Cohomological side : Gysin sequence

$M(\mathcal{A}^H) \subset M(\mathcal{A}\setminus H)$ is a submanifold of real codimension 2

$\Rightarrow H^*(M(\mathcal{A}\setminus H), M(\mathcal{A}); \mathbb{Z}) \simeq H^*(\overset{\text{tubular nbhd}}{T}, \partial T; \mathbb{Z}) \simeq H^*(M(\mathcal{A}^H); \mathbb{Z})[-2]$

So the LES of relative cohomology gives LES (all $\mathbb{Z}$-coefficients) :

$$\cdots \longrightarrow H^p(M(\mathcal{A}\setminus H)) \xrightarrow{\text{restriction}} H^p(M(\mathcal{A})) \xrightarrow{\text{residue}} H^{p-1}(M(\mathcal{A}^H)) \longrightarrow H^{p+1}(M(\mathcal{A}\setminus H)) \longrightarrow \cdots$$

no "-1" since $H = H_1$

$$[\omega_S] \longmapsto \begin{cases} [\omega_{S\setminus H}], & H \in S \\ 0, & \text{otherwise} \end{cases} \quad \text{surjective!}$$

Corollary : LES splits into $H^p(M(\mathcal{A}\setminus H)) \hookrightarrow H^p(M(\mathcal{A})) \twoheadrightarrow H^{p-1}(M(\mathcal{A}^H))$

Algebraic side : direct construction

$$OS(\mathcal{A}) \longrightarrow OS(\mathcal{A}^H) : e_S \longmapsto \begin{cases} e_{S\setminus H}, & H \in S \\ 0, & \text{otherwise} \end{cases}$$

Surjective! graded $\mathbb{Z}$-linear map of degree -1.

well-definedness : $I(\mathcal{A})$ is generated by $\{e_S \wedge \partial e_C \mid S \subset \mathcal{A}, C \text{ circuit}\}$ as $\mathbb{Z}$-mod. (dependent is enough)

If $H \notin C$, then $e_S \wedge \partial e_C \longmapsto \begin{cases} e_{S\setminus H} \wedge \partial e_C, & H \in S \\ 0, & \text{otherwise} \end{cases} \in I(\mathcal{A}^H)$

Otherwise, $e_C = e_1 \wedge e_{C'}$, C' is dependent in $\mathcal{A}^H$ $\Rightarrow e_{C'}, \partial e_{C'} \in I(\mathcal{A}^H)$

$$e_S \wedge \partial e_C = e_S \wedge e_{C'} + (-1)^{|S|} e_1 \wedge e_S \wedge \partial e_{C'} = \begin{cases} e_{S\setminus H} \wedge e_{C'}, & H \in S \\ (-1)^{|S|} \cdot e_S \wedge \partial e_{C'}, & \text{otherwise} \end{cases} \in I(\mathcal{A}^H)$$

Kernel : $e_1 \wedge x + y$, $x \in I(\mathcal{A}^H)$, $y \in \mathrm{Im}(OS(\mathcal{A}\setminus H) \to OS(\mathcal{A}), e_i \mapsto e_i, i \geq 2)$

$\forall C' \subset \mathcal{A}^H$ dependent, $C' \cup \{H\}$ dependent in $\mathcal{A}$ $\Rightarrow e_1 \wedge \partial e_{C'} - e_{C'} \in I(\mathcal{A})$

$\Rightarrow \forall x \in I(\mathcal{A}^H)$, $e_1 \wedge x$ is in $I(\mathcal{A}) + \mathrm{Im}(OS(\mathcal{A}\setminus H) \to OS(\mathcal{A}))$

$\Rightarrow OS^p(\mathcal{A}\setminus H) \longrightarrow OS^p(\mathcal{A}) \longrightarrow OS^{p-1}(\mathcal{A}^H) \longrightarrow 0$ is exact, $\forall p$

Combining two sides together, we obtain a commutative diagram ^of $\mathbb{Z}$-mods^ with exact rows :

$$\begin{tikzcd}
 & OS^p(A \setminus H) \arrow[r, "③"] \arrow[d, "①"'] & OS^p(A) \arrow[r] \arrow[d, "④"'] & OS^{p-1}(A^H) \arrow[r] \arrow[d, "②"'] & 0 \\
0 \arrow[r] & H^p(M(A \setminus H)) \arrow[r] & H^p(M(A)) \arrow[r] & H^{p-1}(M(A^H)) \arrow[r] & 0
\end{tikzcd}$$

①, ② are isomorphism by induction hypothesis.

③ is injective since ④ ∘ ③ is injective

Then ④ is isomorphism by five lemma as $\mathbb{Z}$-mods ! but obviously ring hom $\square$

Cor : $H^*(M(A), \mathbb{Z})$ has no torsion and $H_p(M(A), \mathbb{Z}) \simeq H^p(M(A), \mathbb{Z})^\vee$.

Ex : Prove that $\pi_A(t) = \sum_{p=0}^{\infty} h^p(M(A)) t^p$, $\forall A$ central

Rmk : Standard pattern :

combinatorics $\xrightarrow{\text{determine}}$ (local) topological information $\xrightarrow{\text{recover}}$ invariants

What if "local" is not enough ?

§ 3 Open Problems : Is $H^*(M(\mathcal{A}), \mathbb{L})$ combinatorially determined?

Fix a base field $\mathbb{K}$ ($= \mathbb{C}$ in most cases). $\mathcal{A}$ in $\mathbb{P}^n$

rank-1 local system $\mathbb{L} \xleftrightarrow{1:1}$ representation $\pi(M(\mathcal{A})) \longrightarrow \mathbb{K}^\times$

$$\bigoplus_{H\in\mathcal{A}} \mathbb{Z}\cdot\gamma_H \Big/ \mathbb{Z}\cdot\sum_{H\in\mathcal{A}}\gamma_H = H^1(M(\mathcal{A}),\mathbb{Z})^\vee \simeq H_1(M(\mathcal{A}),\mathbb{Z})$$

$\gamma_H(\omega_{H'}) = \delta_{H,H'}$, local cycle $\oint^H \gamma_H$

i.e. monodromy map $m : \mathcal{A} \longrightarrow \mathbb{K}^\times$ s.t. $\prod_{H\in\mathcal{A}} m(H) = 1$

Q : Are Betti numbers $h^*(M(\mathcal{A}), \mathbb{L}) = \dim_{\mathbb{K}} H^*(M(\mathcal{A}), \mathbb{L})$ determined by $L(\mathcal{A})$ & m ?

Motivation : $f = \prod_{i=1}^r L_i^{d_i}$, L_i linear $\rightsquigarrow$ $\mathcal{A} = \{ H_i = \{L_i = 0\} \}$ in $\mathbb{P}^n$

p : Milnor fiber $F \xrightarrow{d = (\sum_{i=1}^r d_i) : 1} M(\mathcal{A})$

$$\underline{\mathbb{C}} \rightsquigarrow p_* \underline{\mathbb{C}} = \bigoplus_{j=0}^{d-1} \mathbb{L}_{\chi_d^j}$$

$$\chi_d^j : \mathcal{A} \longrightarrow \mathbb{K}^\times , \quad \chi_d^j(H_i) = e^{\frac{d_i}{d}\cdot 2\pi\sqrt{-1}\cdot j}$$

$$\rho^* = T^{-1} \curvearrowright H^*(F, \underline{\mathbb{C}}) \cong H^*(M(\mathcal{A}), p_*\underline{\mathbb{C}}) = \bigoplus_{j=0}^{d-1} H^*(M(\mathcal{A}), \mathbb{L}_{\chi_d^j})$$

compatible

Main attention

Difference from the constant case : ① Deletion-restriction fails

② Absence of (local) generators $\rightsquigarrow$ vanishing in most cases

Known results & Main conjecture $\mathbb{K} = \mathbb{C}$

Main vanishing results involve resonant dense flats.

Def: A flat $\mathcal{F}$ is called dense if $\mathcal{A}_{\mathcal{F}} \neq \mathcal{A}_{\mathcal{F}_1} \cup \mathcal{A}_{\mathcal{F}_2}$, $\forall$ flats $\mathcal{F}_1, \mathcal{F}_2 < \mathcal{F}$ s.t. $\mathcal{F}_1 \wedge \mathcal{F}_2 = \mathcal{F}$

A dense flat $\mathcal{F}$ is called resonant w.r.t. $\mathbb{L}$ if $\prod_{H \in \mathcal{F}} m(H) = 1$

Thm 1: If $\exists H \in \mathcal{A}$ s.t. no resonant dense flat contains H

then $h^p(M(\mathcal{A}), \mathbb{L}) = \begin{cases} 0 & , p < n \\ \pi_{\mathcal{A}}(1) & , p = n \end{cases}$ "Non resonant condition"

Cor 2: $h^p(M(\mathcal{A}), \mathbb{L}) = 0$, $\forall p < n$ for generic $\mathcal{A}$ & nonconstant $\mathbb{L}$

Thm 3 (Cohen 1998) $h^p(M(\mathcal{A}), \mathbb{L}) \leq h^p(M(\mathcal{A}), \mathbb{C})$, $\forall p$

Fact 4: Let $H \subset \mathbb{P}^n$ be a generic hyperplane w.r.t. $\mathcal{A}$.

Then $\mathcal{B} = \{ H' \cap H \mid H' \in \mathcal{A} \}$ is a hyperplane arrangement in $H \simeq \mathbb{P}^{n-1}$.

and $H^p(M(\mathcal{A}), \mathbb{L}) \simeq H^p(M(\mathcal{B}), \mathbb{L}|_{M(\mathcal{B})})$, $\forall p \leq n-2$

from Milnor fiber

Simplest case: const monodromy (d-th root of unity) + $h^1 \xrightarrow{\text{Fact 4}}$ line arrangement

In this case: $L(\mathcal{A}) \longleftrightarrow \mathcal{A}_p$, $\forall p \in L_2(\mathcal{A}) = \{\text{pts}\}$, resonant $\Leftrightarrow$ $d \mid \underbrace{\text{mult}(p) \geq 3}_{\text{dense}}$

Thm 5: $h^1(M(\mathcal{A}), \mathbb{L}) \leq \sum_{p \in L, d \mid \text{mult}(p)} (\text{mult}(p) - 2)$ for any $l \in \mathcal{A}$

all cycles come from resonant pts.

Fact 6: All known examples for $h^1 > 0$ admit highly symmetric structures,

which only allow d to be 2, 3 or 4.

Conjecture: (Papadima & Suciu, 2017) $h^1(M(\mathcal{A}), \mathbb{L}) = 0$, $\forall d = p^s > 5$

prove this holds when $\text{mult}(p) = 2$ or 3, $\forall p$.