Arithmeticity of monodromy representations of braid groups

- Basic definitions

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Basic definitions

We consider the following base space:

- $S = \mathbb{C}^{n+1} \setminus A$

$A = \{(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_i = z_j \text{ for some } i,j \in \{1, \cdots, n+1\}\}$

There is a natural smooth family:

$\pi: \quad X_{\underline{k}} \longrightarrow S, \ \underline{k} = (k_1, \cdots, k_{n+1}) \quad (k_i, d) = 1 \quad 1 \le i \le n+1$

- fiber of π at $(a_1, \cdots, a_{n+1})$ is the curve $X_{\underline{k}, \underline{a}}$

$$\{(y, x) \mid y^d = (x-a_1)^{k_1} \cdots (x - a_{n+1})^{k_{n+1}}, \ y \neq 0, \ (k_i, d) = 1\}$$

- $X = \{(y, x, \underline{a}) \in \mathbb{C}^2 \times \mathbb{C}^{n+1} \mid y^d = \prod (x-a_i)^{k_i}, \ y \neq 0, \ a_i \neq a_j \ \forall i,j \in \{1, \cdots, n+1\}\}$

We have naturally the monodromy representation

$$\rho_{\underline{k}}: \pi_1(S) \longrightarrow GL(H_1(X_{\underline{k}, \underline{a}}, \mathbb{Z}))$$

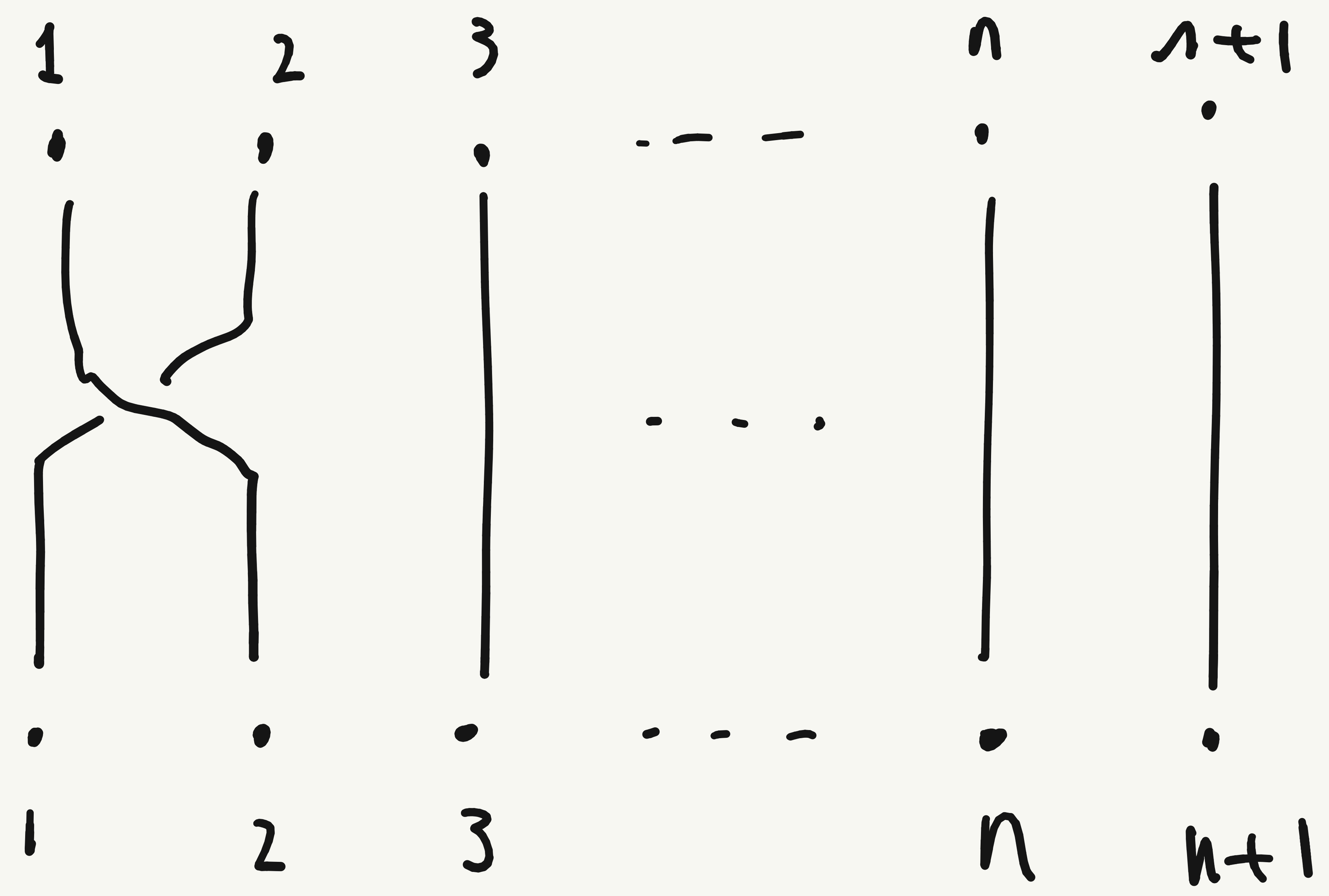
We identify $\pi_1(S)$ with the pure braid group P_{n+1}:

Def: The braid group B_{n+1} on $(n+1)$ strands is the free group on n generators $s_1, \cdots, s_n$ modulo the relations

$$s_j s_k = s_k s_j \quad (|j-k| \geq 2)$$

$$s_j s_k s_j = s_k s_j s_k \quad (|j-k| = 1)$$

s_1 :



Note that $S_{n+1} = \langle t_1, \cdots, t_n \rangle$ modulo the relation

$$t_j t_k = t_k t_j \qquad (|j-k| \geq 2)$$

$$t_j t_k t_j = t_k t_j t_k \qquad (|j-k| = 1)$$

$$t_j^2 = 1 \qquad (j = 1, \cdots, n)$$

Then there is a natural quotient map:

$$B_{n+1} \longrightarrow S_{n+1} \qquad s_i \mapsto t_i = (i\ \ i+1)$$

with kernel P_{n+1}.

For each loop $\gamma(t) = (z_1(t), \cdots, z_{n+1}(t))$ in $\pi_1(S)$

$\{z_1(t)\} \cup \cdots \cup \{z_{n+1}(t)\}$ is a pure braid in $\mathbb{C} \times [0,1]$. So,

$\pi_1(S) \cong P_{n+1}$. $\left(\pi_1(S/S_{n+1}) \cong B_{n+1}\right)$

- <u>Def</u>: A subgroup $\Gamma \subset GL_N(\mathbb{Z})$ is an <u>arithmetic group</u> if Γ is of finite index in $G(\mathbb{Z})$. Here G is the Zariski closure of Γ in GL_N. (Note that G is not naturally a $\mathbb{Z}$-algebraic group, so we have to choose a $\mathbb{Z}$-model of G. Each $G(\mathbb{Z})$ from different $\mathbb{Z}$-models is commensurable with each other.)

<u>Question</u>: Is $\rho_{\underline{k}}(\pi_1(S)) = \rho_{\underline{k}}(P_{n+1})$ an arithmetic group in $GL(H_1(X_{\underline{k},\underline{a}}, \mathbb{Z}))$?

<u>Thm</u>. Suppose $d \geq 2$ and $n \geq 1$, $1 \leq k_i \leq d-1$, $\gcd(k_i, d) = 1$. Suppose that $n \geq 2d$. Then $\rho_{\underline{k}}(P_{n+1})$, $\rho_{\underline{k}}(P_{n+1})$ are arithmetic subgroups of $GL(H_1(X_{\underline{k},\underline{a}}, \mathbb{Z}))$, $GL(H_1(X^*_{\underline{k},\underline{a}}, \mathbb{Z}))$ respectively.

<u>Rmk</u>: When $n < 2d$, the arithmeticity not always holds.

E.g. $y^{18} = (x-b_1)(x-b_2)(x-b_3)(x-b_4)$. By Deligne - Mostow, the monodromy group is not arithmetic.

- <u>Monodromy representation</u>

From now on, we only consider $k_1 = k_2 = \cdots = k_{n+1} = 1$ for simplicity.

$\mathbb{Z}/d\mathbb{Z} \to X_{\underline{a}} \to \mathbb{C} \setminus \{a_1, \cdots, a_{n+1}\}$ cyclic covering, induces

$$1 \to \pi_1(X_a) \to F_{n+1} \to \mathbb{Z}/d\mathbb{Z} \to 1$$
$$x_i \mapsto [1]$$

The inclusion $X_a \hookrightarrow X_a^*$ induces

$\pi_1(X_a) \twoheadrightarrow \pi_1(X_a^*)$ by quotient out the loops around $a_1, \cdots, a_{n+1}$ and ∞.

These maps can be lifted to the family version:

Let $\mathcal{F} = \{(y, x, \underline{a})\ \mathbb{C}^* \times \mathbb{C} \times S \mid y = (x-a_1)\cdots(x-a_n)\}$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ X^* \arrow[dr] & \, \arrow[d] & \mathcal{F} \arrow[dl] \\ & S & \end{tikzcd}$$

So, we may first analyze $\pi_1(S)$ acting on F_{n+1} at (by parallel transport) first.

- Recall that $\pi_1(S/S_{n+1}) = B_{n+1}$ and the above families are equivariant w.r.t. the action of S_{n+1}. (isoms. fiberwise)

$\Rightarrow$

$$\begin{tikzcd}
 & X/S_{n+1} \arrow[dl] \arrow[dr] \arrow[dd] & \\
X^*/S_{n+1} \arrow[dr] & & \mathcal{F}/S_{n+1} \arrow[dl] \\
 & S/S_{n+1} &
\end{tikzcd}$$

That is, the monodromy action on $\pi_1(X_{\underline{a}})$, $\pi_1(X^*_{\underline{a}})$ $(H_1(X_{\underline{a}}), H_1(X^*_{\underline{a}}))$ and F_{n+1} can be extended to B_{n+1}.

- $B_{n+1} \curvearrowright F_{n+1}$ as follows:

$$s_i(x_j) = x_j \qquad j \neq i, i+1.$$

$$s_i(x_i) = x_{i+1}$$

$$s_i(x_{i+1}) = x_{i+1}^{-1} x_i x_{i+1}$$

- Denote by $K_{n+1}(d) = \pi_1(X_{\underline{a}})$

$$1 \longrightarrow K_{n+1}(d) \longrightarrow F_{n+1} \longrightarrow \mathbb{Z}/d\mathbb{Z} \to 1$$

$K_{n+1}(d)$ generated by x_1^d, $y_i = x_i^{-1} x_{i+1}$, $x_1^j y_i x_1^{-j}$ $0 \le j \le d-1$.

It is a free group of rank $1 + dn$.

By quotient out $[K_{n+1}(d), K_{n+1}(d)]$, the conjugation action of F_{n+1} on $K_{n+1}(d)^{ab}$ descends to $\mathbb{Z}/d\mathbb{Z}$.

So $K_{n+1}(d)^{ab}$ is a $\mathbb{Z}[q]/(q^d-1)$-module : a direct sum of free module generated by y_i and trivial module generated by x_1^d i.e. $K_{n+1}(d)=\bigoplus_{i=1}^{n}\left(\mathbb{Z}[q]/(q^d-1)\cdot y_i\right)\oplus\left(\mathbb{Z}\cdot x_1^d\right)$

By carefully analysis the action of q and B_{n+1} on each y_i and x_1^d, we obtain:

① The action of B_{n+1} commutes with $\mathbb{Z}[q]/(q^d-1)$ and B_{n+1} acts on x_1^d trivially. So, $B_{n+1}\subset GL(n,\mathbb{Z}[q]/(q^d-1))\oplus\{1\}$.

② Replace $\{y_1,\cdots,y_n\}$ by another basis $\{e_1,\cdots,e_n\}$, $y_i=q^ie_i$.

$S_i(e_j)=e_j\ (|i-j|\geq 2)$, $S_i(e_{i-1})=e_{i-1}+q\,e_i$

$S_i(e_i)=-q\,e_i$, $S_i(e_{i+1})=e_{i+1}+e_i$.

Rmk: Action of q comes from geometry: $y\mapsto \exp(\frac{2\pi i}{d})y$

From this, q clearly commutes with the action of B_{n+1}.

- $\mathbb{Z}[q]/(q^d-1) \cong \bigoplus_{e|d} \mathbb{Z}[q]/\Phi_e(q)$ ← cyclotomic polynomial.

Now $K_{n+1}(d) = \bigoplus_{q|d} \bigoplus_{i=1}^{n} (\mathbb{Z}[q]/\Phi_e(q) \cdot y_i) \oplus (\mathbb{Z} x_i^d)$

$= (\bigoplus_{e|d} \rho_n(e)) \oplus \rho_{trivial}$

Here $\rho_n(e): B_{n+1} \longrightarrow GL(n, \mathbb{Z}[\zeta_e])$, the same rule as ②

is called the <u>Burau representation</u> at e-th root of unity.

- <u>Unipotent elements and arithmeticity</u>

Note that $B_{n+1}/P_{n+1} = S_{n+1}$, a finite group. So.

the arithmeticity of P_{n+1} and B_{n+1} are the same.

We first consider the arithmeticity at each

factor $\rho_n(e)$: (ignore the trivial part, no effect).

$e=1$ $\rho_n(1)$ factor through S_{n+1} finite group.

Doesn't change arithmeticity. ignore it.

$e=2$ $\rho_n(2) \subseteq Sp(n,\mathbb{Z})$. By a result of A'Campo.

(it is $\Gamma(2)$, congurence subgroup)
it is an arithmetic subgroup in $Sp(n,\mathbb{Z})$. We omit details.

For $e \geq 3$, we have the following main theorem due to Venkataramana:

<u>Thm</u>: If $d \geq 3$ and $n \geq 2d$, then the image $\rho_n^{(d)}(B_{n+1})$ of the Braid group in $GL(n, \mathbb{Z}[q]/\Phi_d(q))$ is an arithmetic group.

<u>Rmk</u>: If $\Gamma \subseteq GL(n, \mathcal{O}_k)$, here $\mathcal{O}_k$ is an algebraic integer ring. Say Γ is arithmetic if Γ is of finite index in $\Gamma(\mathcal{O}_k)$.

To deal with all roots of unity universally, we introduce a <u>formal Burau rep.</u> $\rho_n : B_{n+1} \longrightarrow GL(n, \mathbb{Z}[q, q^{-1}])$ by

$$s_i(e_j) = e_j \ (|i-j| \geq 2), \quad s_i(e_{i-1}) = e_{i-1} + q e_i$$

$$s_i(e_i) = -q e_i, \quad s_i(e_{i+1}) = e_{i+1} + e_i$$

There is a natural involution $\mathbb{Z}[q, q^{-1}] \longrightarrow \mathbb{Z}[q, q^{-1}]$,

$$q \longmapsto q^{-1}$$

so that the following form:

$$h = \begin{pmatrix} \frac{(q+1)^2}{q} & -(1+q) & 0 & \cdots & \cdots \\ -(1+q^{-1}) & \frac{(q+1)^2}{q} & -(1+q) & \cdots & \cdots \\ 0 & -(1+q^{-1}) & \frac{(q+1)^2}{q} & -(1+q) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \qquad \det h = \left(\frac{q+1}{q}\right)^n \left(\frac{q^{n+1}-1}{q-1}\right)$$

is hermitian. B_{n+1} preserves such form. actually:

$S_k(x) = x - \frac{q h(x, e_k)}{q+1} e_k$. ($\{e_1, \cdots, e_n\}$ standard basis)

Specialize at $q = \zeta_d$, we get the usual Burau rep.

So, $\rho_n(d)(B_{n+1}) \subseteq U(h, K)$, $K = \mathbb{Q}(\cos(\frac{2\pi}{d}))$.

A standard argument implies:

<u>Prop</u>: • When $n \not\equiv -1 \pmod d$, h is nondegenerated and $\rho_n(d)$ is irreducible

• When $n \equiv -1 \pmod d$, h is degenerated and its radical spanned by $v = e_1 + \frac{q^2-1}{q-1} e_2 + \cdots + \frac{q^{kd-1}-1}{q-1} e_{kd-1}$

($\langle v \rangle$ fixed by B_{n+1})

$\rho_n(d)/\langle v \rangle$ is irreducible and $\rho_n(d)/\langle v \rangle \Big|_{B_n} = \rho_{n-1}(d)$.

The arithmeticity is based on the following theorem:

<u>Thm</u> (Bass, Milnor, Serre, Tits, Vaserstein, Venkataramana, Raghunathan).

Let G/K be an algebraic group s.t. $G(\mathbb{C})$ is simple. Assume
K number field.

that K-rank$(G) \geq 2$ ($\exists$ maximum split K-torus G_m^r of rank r).

Let P, P^- be two opposite parabolic subgrps containing a maximal K-split torus, and U, U^- their unipotent radicals

For any integer $N \geq 1$, the group $\Delta_N(P^\pm)$ generated by

N-th powers of $U(\mathcal{O}_K)$ and $U^-(\mathcal{O}_K)$ is of finite index in

$G(\mathcal{O}_K)$, thus is arithmetic.

- So we only need to find an invariant flag and find enough unipotent elements in $\rho_n(d)(B_{n+1})$.

- The key idea is to construct from the degenerated rep. ($n \equiv -1 \pmod d$ case) The radical $\langle v \rangle$ plays a role of a vanishing cycle.

$\Delta = (S_1 \cdots S_n)(S_1 \cdots S_{n-1}) \cdots (S_1 S_2)(S_1) \Rightarrow \Delta^2$ in the center of B_{n+1} (also in P_{n+1}). $\Delta' = (S_2 \cdots S_n)(S_2 \cdots S_{n-1}) \cdots (S_2) \Rightarrow \Delta'^2$ in the center of $\langle S_2, \cdots, S_n \rangle \cong B_n \subseteq B_{n+1}$. Note that $B_n \curvearrowright \mathrm{Span}_A\{e_2 \cdots e_n\}$ $\left(A = \mathbb{Z}[\zeta_d]\right)$ as $\rho_{n-1}(d)$, irreducibly ($n-1 \not\equiv -1 \pmod d$). So Δ'^2 acts on $\mathrm{Span}_A\{e_2, \cdots, e_n\}$ as a scalar (i.e. B_n-module homomorphism)

Consider the flag $\mathcal{F}: Av \subset Av + \underbrace{Ae_2 + \cdots + Ae_{n-1}}_{=X} \subset A^n$.

$R(\mathcal{F})$: unipotent subgroups of $U(h)$ preserving the flag and acting trivially on the successive quotient

Direct computation shows $a \in R(\mathcal{F})$ is of the form

$$\begin{pmatrix} 1 & a_2 \cdots a_{n-1} & c \\ & I_{n-1} & \end{pmatrix}$$

So $R(\mathcal{F}) \cong \mathrm{Hom}(X, Av) \rtimes \mathbb{G}_a(A)$.

Now consider $N = \langle huh^{-1} \mid h \in \langle s_2, \ldots, s_n \rangle = B_n \rangle$, $u = [\Delta'^2, s_1]$

$\Rightarrow N \subseteq R(\mathcal{F})$ and N acts on $\mathrm{Hom}(X, Av)$ as B_n acts on X.

From this one can show $N \subseteq R(\mathcal{F})$ is of finite index

So we construct enough unipotent elements in this case.

- For $n = kd$ $(n-1 \equiv -1)$ $\quad B_n = \langle s_1 \cdots s_n \rangle$, $B_n' = \langle s_2 \cdots s_n \rangle$

will give v, v'. One can construct unipotent elements similarly. The isotropic plane $\mathrm{Span}\{v, v'\}$ ensuring ($n \geq 2d$ is needed).

K-rank$(U(h)) \geq 2$.

- Induction shows the remained case

The following thm (based on Mostow rigidity) glues arithmeticity of each factors to a global property and the main thm follows

<u>Thm</u>: Suppose K_e number fields for each element $e \in X$. $|X| < +\infty$. Suppose for each G_e, $\mathfrak{g}_e(\mathbb{C})$ simple and ∞-rank$(G_e) = \sum_{v \text{ arch}} K_v\text{-rank}(G_e) \geq 2$. Suppose $\Gamma \subset \prod_{e \in X} G_e(\mathcal{O}_e)$ is s.t. Γ proj. to each $G_e(\mathcal{O}_e)$ of finite index. And either $K_e \not\cong K_f$ $e \neq f$ or $G_e \not\cong G_f$. Then $\Gamma \subset \prod_{e \in X} G_e(\mathcal{O}_e)$ arithmetic.

- For $H_1(X^*, \mathbb{Z})$. it is a quotient rep. of $H_1(X, \mathbb{Z})$, arith. follows
- For $(k_1, \cdots k_{n+1}) \neq (1, \cdots, 1)$, consider Gassner rep. to handle these weights $(\frac{k_i}{d})$.

<u>Open problem</u>: Arithmeticity of hypergeometric groups

Note that B_{n+1} generated by $(s_1 \cdots s_n)$ & (s_n^{-1}). So, the Burau rep. is a special case of monodromy rep. of hypergeometric group. And these thm shows the arithmeticity of a limited family of hypergeometric groups.

Is there any techniques here useful for the general case?