Irrationality of $\zeta(3)$ and a K3 family

- Apéry's proof of $\zeta(3)$ irrationality

- Beukers' geometric interpretation via K3 family

- Similar phenomena and some remarks

- Open problems

• Apéry's proof of $\zeta(3)$ irrationality

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.2020569031595\ldots$$

Apéry constructed

$$\{A_n\}_{n\geq 0} = \{1, 5, 73, 1445, 33001, \ldots\}$$

$$\{B_n\}_{n\geq 0} = \left\{0, 6, \frac{351}{4}, \frac{62531}{36}, \frac{11424695}{288}, \ldots\right\}$$

① $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$ are solutions of

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)\, u_n + n^3 u_{n-1} = 0$$

with initial conditions $(A_0, A_1) = (1, 5)$ and $(B_0, B_1) = (0, 6)$.

② $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z}$. $n \geq 0$.

③ $B_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2$ $d_n^3 B_n \in \mathbb{Z}$. $d_n = \text{l.c.m.}\{1, 2, \ldots, n\}$, $n \geq 0$

④ $\left|\zeta(3) - \frac{B_n}{A_n}\right| \sim (\sqrt{2}-1)^{8n}$ $d_n \sim e^n$ $A_n \sim (\sqrt{2}-1)^{-4n}$

If $\zeta(3) = \frac{p}{q}$

$$\Rightarrow 0 < q A_n d_n^3 \left|\zeta(3) - \frac{B_n}{A_n}\right| \sim c \cdot \frac{e^n}{(\sqrt{2}+1)^{4n}} < 1$$

Cannot be possible.

Idea of Proof:

- Write $A(t) = \sum A_n t^n$ as

$$\frac{1}{(2\pi i)^3} \int_S \frac{dX\,dY\,dZ}{1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z)} \qquad (1)$$

$\downarrow$

- Use Poincaré Residue to express (1) as period integral on S_t: $1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z) = 0$

$\downarrow$

- Show that S_t birational eq. to K3. X_t.

$\downarrow$

- Use branched locus to show X_t is not constant

Construct elliptic fibration on X_t by running the minimal resolution. Use elliptic fibration to estimate the Picard number.

- Beukers' geometric interpretation via K3 family

- Let $\mathcal{A}(t) = \sum_0^\infty A_n t^n \qquad \mathcal{B}(t) = \sum_0^\infty B_n t^n$

$$\mathcal{R}(t) = \mathcal{A}(t)\zeta(3) - \mathcal{B}(t).$$

By ①, $L\mathcal{A}(t) = 0 \qquad L\mathcal{B}(t) = 5 \qquad L\mathcal{R}(t) = -5$

$$L = (t^4 - 34t^3 + t^2)\left(\frac{d}{dt}\right)^3 + (6t^3 - 153t^2 + 3t)\left(\frac{d}{dt}\right)^2$$
$$+ (7t^2 - 112t + 1)\frac{d}{dt} + t - 5 .$$

- Let $P_n(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n (x^n(1-x))^n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k$

$$\Rightarrow \quad A_n = \sum \binom{n}{k}^2 \binom{n+k}{k}^2 = \frac{1}{2\pi i}\int_{|x|=1} \frac{1}{x} P_n(x) P_n\left(\frac{1}{x}\right) dx$$

$$\underset{\substack{\text{integral by parts} \\ \text{(see [2])}}}{\overset{\text{change variable}}{=}} \int_S \frac{X^n(1-X)^n Y^n(1-Y)^n Z(1-Z)^n}{(1-(1-XY)Z)^{n+1}} \, dX\, dY\, dZ$$

S: real dim 3 area.

$$\Rightarrow \quad \mathcal{A}(t) = \frac{1}{(2\pi i)^3}\int_S \frac{dX\, dY\, dZ}{1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z)}$$

Let $S_t = \{1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z) = 0\} \subseteq \mathbb{C}^3$

$$\Omega_t = \frac{dX\,dY\,dZ}{1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z)}$$

$\Rightarrow \quad S \subseteq \mathbb{C}^3 \setminus S_t$.

- $S = \partial R \quad R$: 4- chains in $\mathbb{C}^3$

  Assume R in general position & $R \cap S_t = \gamma$

$$\begin{cases} \gamma: \text{ 2- cycle in } S_t \\ T_\varepsilon(\gamma) = \{x \in R \mid d(x, \gamma) < \varepsilon\}. \\ \tau_\varepsilon(\gamma) = \partial T_\varepsilon(\gamma) \end{cases}$$

$\Rightarrow \partial(R - T_\varepsilon(\gamma)) = S - \tau_\varepsilon(\gamma)$

$\Rightarrow \int_S \Omega_t = \int_{\tau_\varepsilon(\gamma)} \Omega_t$

$\xrightarrow[\text{residue}]{\text{Poincaré}}$ $$\int_{\tau_\varepsilon(\gamma)} \Omega_t = 2\pi i \int_\gamma \frac{dX \wedge dZ}{\frac{\partial}{\partial Y}(1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z))}$$

$$= 2\pi i \int_\gamma \frac{dX \wedge dZ}{XZ(1-t(1-X)(1-Z)(1-2Y))}$$

So far, we obtain that $A(t)$ is the integration

of a two form $\omega_t := \dfrac{dX \wedge dZ}{XZ(1-t(1-X)(1-Z)(1-ZY))}$ on a 2-cycle

γ of a family of surface S_t defined by

$$1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z) = 0$$

relation to the order of decre?

• Main theorem([1]): Let $t \notin \{0, 1, (\sqrt{2} \pm 1)^4, \infty\}$. Then,

i) The algebraic surface S_t given by

$$1-(1-XY)Z - tXYZ(1-X)(1-Y)(1-Z) = 0$$

is birationally equivalent to a K3 surface X_t.

ii) The form ω_t, up to a scalar, is the unique holomorphic 2-form on X_t.

iii) The Picard number satisfies $\rho(X_t) \geq 19$ and equality holds for all but countably many t.

(two K3 (minimal) with different ρ cannot be bir. equi. (bir. eq. is iso))

Proof: • The affine equation

$$1-(1-XY)Z - t(XYZ)(1-X)(1-Y)(1-Z) = 0$$

becomes

$$z^2 = 4tuv(1-u)^2(1-v)^2 + (u(1-v)t - v(1-u))^2 \quad (\star)$$

after birational substitutions

And ω_t transforms to

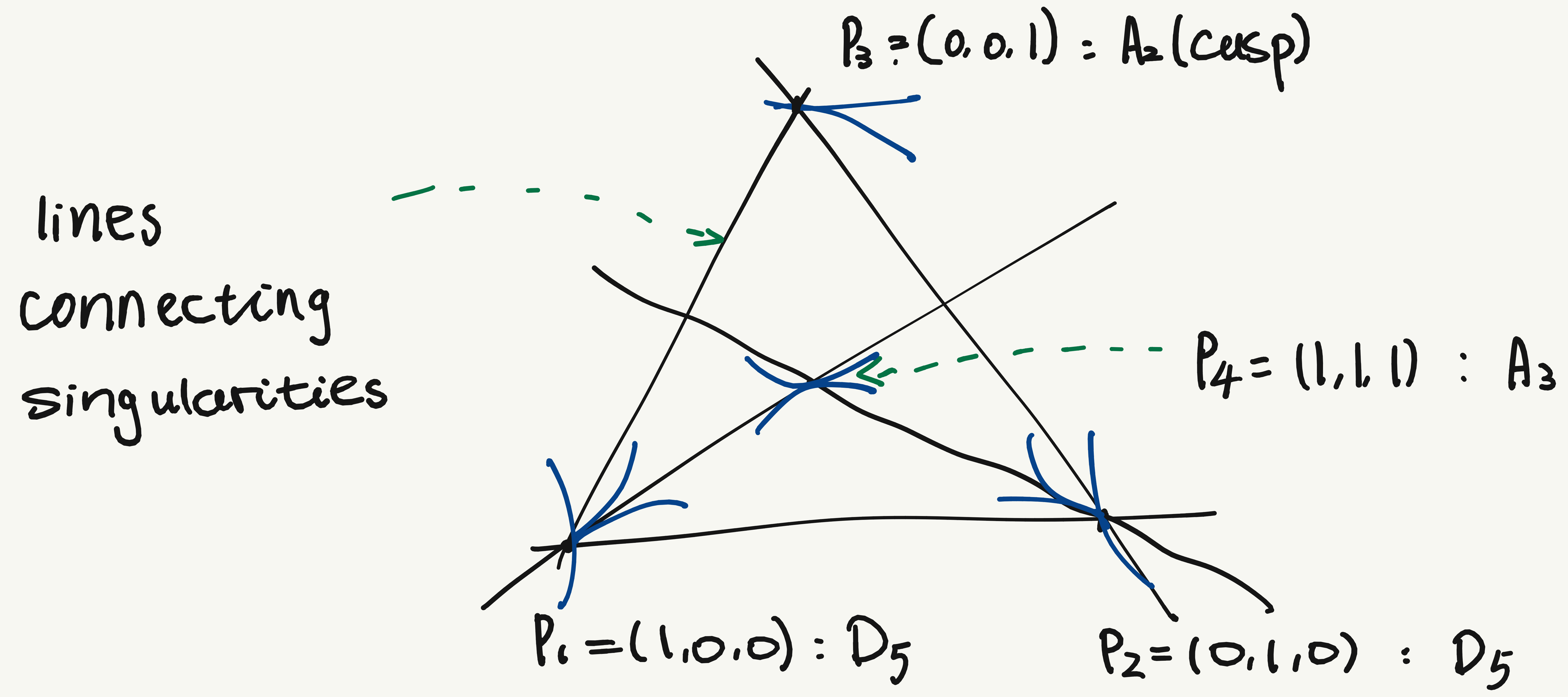
$$\omega_t = \frac{du \wedge dv}{z}.$$

So the surface S_t is birationally equivalent to a double cover $\overline{S}_t$ of $\mathbb{P}^2$ branched along a sextic curve C_t (always singular) with defining equation

$$4tuvw(w-u)^2(w-v)^2 + w^2(u(w-v)t - v(w-u))^2 = 0.$$

- Direct computation shows: for $t \notin \{0, 1, \infty, (\sqrt{2} \pm 1)^4\}$

C_t has only 4 simple singularities:



Singularities of C_t in $\mathbb{P}^2$.

And C_t is birationally equivalent to a elliptic curve (we omit the details in [Beukers].)

Apply the a theorem in complex surface (in [III. thm 7.2. Barth.7]). We know the minimal resolution X_t of $\overline{S_t}$ is a K3 surface ("simple singularies" is important). So (i) & (ii) is OK.

• To compute the Picard number, we apply the following lemma ([Cor 1.5, Shioda. 6]):

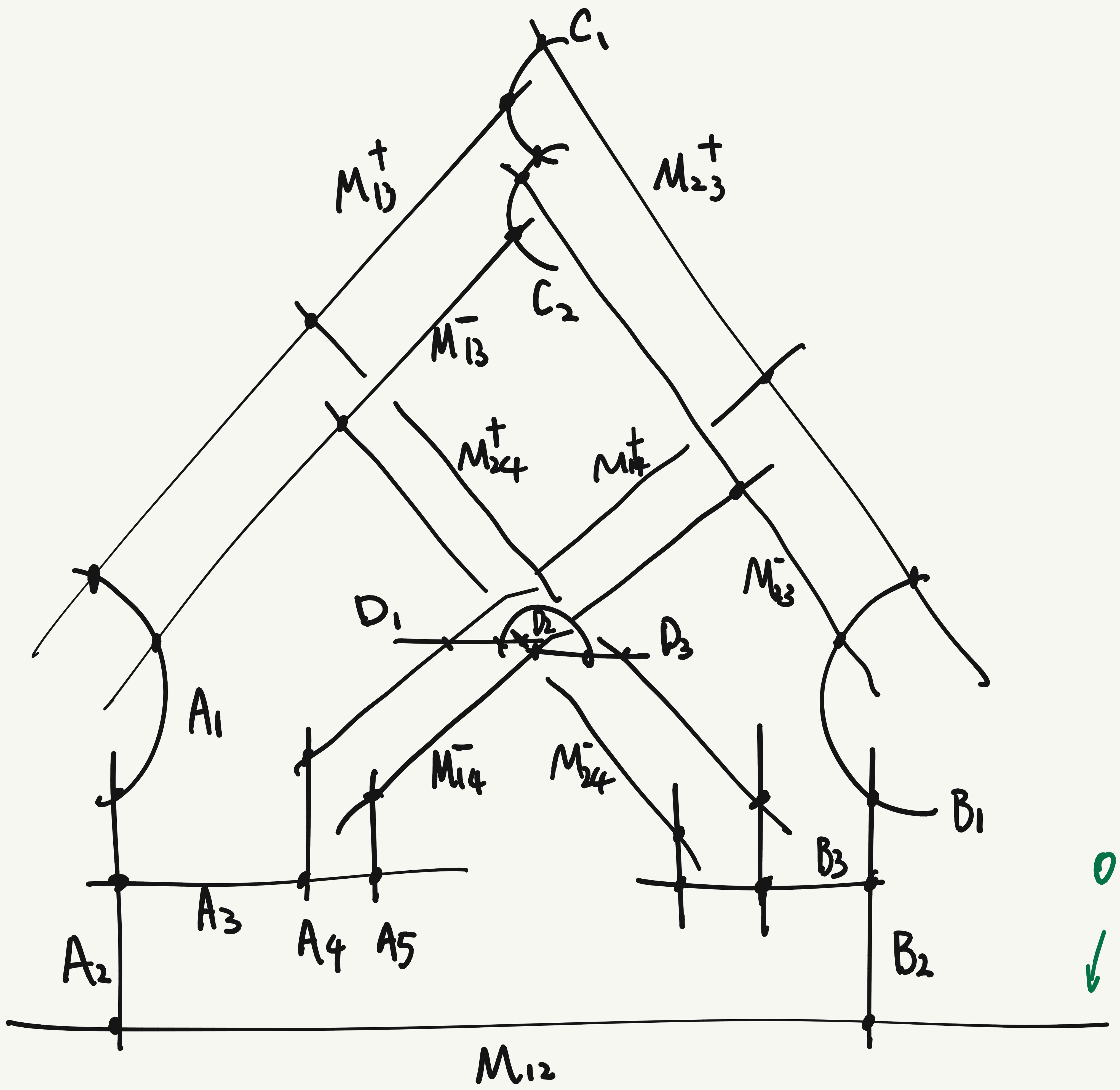
<u>Lem</u>: Let $\pi : B \to \Delta$ be a elliptic surface over a curve Δ with function field K. Then

$\mathrm{rank}_{\mathbb{Z}}(E(K))$ numbers of irr. components

$$\rho(B) = r + 2 + \sum_{\lambda \in \Sigma} (m_\lambda - 1).$$

singular locus

To construct the elliptic fibration, we run the minimal resolution of $\overline{S_t}$ by blow-up the singularities of C_t in $\mathbb{P}^2$. We have



only one. intersects the branched locus A_2

A-curves give P_1 B-curves give P_2

C_1, C_2 give P_3 D_1, D_2, D_3 give P_4.

$\{M_{ij}^{\pm}\}$ preimages of lines connecting P_i, P_j in X_t

The above graph draws some curves in X_t.

actually a fibration. as we blow up P_2, no base points.

- Use elliptic pencil on X_t coming from the lines through P_2. On X_t these lines have inverse images that are elliptic curves all meeting B_2 in exactly one point.

intersect at P_2 with multi 5, so another 1 pt in C_t

(generically)

$$g(C) = 1 + \frac{1}{2}\left(\deg(\pi)(g(\ell)-1) + \sum(e_p - 1)\right)$$
$$= 1 + \frac{1}{2}\left(2\cdot(-1) + (2-1) + (2-1)\right) = 1$$

B_2 another pt

① For generic t, $m_{13}^{\pm}$, $m_{14}^{\pm}$: sections of the fibration have infinite order $\Rightarrow$ $r \geq 1$.

② Some reduced fibers

- M_{12}, A_1, A_2, A_3, A_4, A_5
- B_1, M_{23}^{+}, M_{23}^{-}, C_1, C_2
- B_3, B_4, B_5, M_{24}^{-}, M_{24}^{+}, D_1, D_2, D_3

$\Rightarrow \rho(X_t) \geq 1+2+5+4+7=19.$

The family is not constant, so $\rho(X_t)$ cannot
(j-invariant of elliptic model of C_t varies w.r.t. t)
always ≥ 20. (discrete family by Global Torelli).

- Similar phenomena and some remarks

① Use the similar method, Apéry prove the irrationality of $\zeta(2)$. $\{a_n\}$ & $\{b_n\}$ related to $\zeta(2)$ come from a family of elliptic curves.

② The series $A(t)$ is a solution of $Ly=0$, which is called a Picard-Fuchs function constructed in general as follows:

Given a $\mathbb{P}^1$ family $\mathcal{X} \to \mathbb{P}^1$ of varieties with a local parameter t. Choose a varying closed form

$\omega(t)$ and a constant cycle γ. Then the period integral

$$f(t) = \int_\gamma \omega(t)$$

is a function on $\mathbb{P}^1$ (may have poles).

The derivation $\frac{d}{dt}\omega(t)$ is still a closed form and

$$f'(t) = \frac{d}{dt}\int_\gamma \omega(t) = \int_\gamma \frac{d}{dt}\omega(t)$$

The cohomology has finite dimension, so $\{(\frac{d}{dt})^k \omega(t)\}_{1\leq k\leq n}$

is linearly dependent for some n $\Rightarrow$ $\exists$ $\{C_k(t)\}_{1\leq k\leq n}$

$$\sum_{k=1}^{n} C_k(t)\, f^{(k)}(t) = 0.$$

This defines the Picard-Fuchs function of the period integral $f(t)$.

The Gauss-Manin connection ∇ of the family $\pi: \mathcal{X} \to \mathbb{P}^1$ is a generalization of the Picard-Fuchs equation.

③. The Apéry numbers A_n can be extended to A_x, $x \in \mathbb{C}$ by

$$A_x = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2 ,$$

which is a holomorphic function on $\mathbb{C}$ with $A_x = A_{-x-1}$ and

$$(x+1)^3 A_{x+1} - (34x^3 + 51x^2 + 27x + 5) A_x + x^3 A_{x-1} = \frac{8}{\pi^2}(2x+1)\sin^2 \pi x.$$

<u>Prop</u> ([4]). The value of the function A_n at its point of symmetry is given by

$$A_{-\frac{1}{2}} = \frac{16}{\pi^2} L(f_{4.8}, 2) ,$$

where $f_{4.8}$ is the normalized Hecke eigenform of weight 4, level 8.

This identities can be proved by complex analysis. But it can be expected by the idea of motives as follows :

1) $A_{-\frac{1}{2}}$ can be realized by an algebraic threefold X. a double covering branched along the K3.

2) $\#(X(\mathbb{F}_p)) = p - \gamma_p$

γ_p : the Fourier coefficient of $f_{4,8}$

$\Rightarrow$ The L function of $f_{4,8}$ must be a factor of the ζ-function of X.

By the Tate conjecture, the motive of $f_{4,8}$ (Galois rep. of $f_{4,8}$). must be a part of the cohomology of X. This in turn hints that the special values of $L(f_{4,8})$ should appear as periods of X.

$$\begin{tikzcd}
 & \text{Variety } X \arrow[dl, <->, "\text{integral}", "{H_{dR}(X),\ H_B(X)}"'] \arrow[dr, <->, "H_{et}(X)", "{\text{point counting}}"'] & \\
\text{periods} \arrow[rr, <->, "\text{numerical}", "\text{identity}"'] & & \text{L-functions of modular forms}
\end{tikzcd}$$

- Open problems

① Let $\{a_n\}$ be the Apéry numbers of $\zeta(2)$. Find some extension a_x of a_n. Find some relation of special values of a_x and special values of the L-function of some modular forms,

② ([5]) For the Apéry-like recursions (generalize to $\zeta(2)$) $(A,B,C)=(11,3,-1)$

$$(n+1)^2 u_{n+1} - (An^2+An+B)u_n + Cn^2 u_{n-1} = 0.$$

Zagier tested when the recursion has integer solutions and constructed geometric realizations. (for 100 milion triples (A,B,C))

Question: 1). Find a condition of (A,B,C) to have integer solutions and produce geometric realization of these solutions.

2). Generalized to deg 3 Apéry-like recursions (generalize the recursion for A_n of $\zeta(3)$).

3) Extend these integer solutions $\{a_n\}$ to $\{a_x\}$ $x \in \mathbb{C}$ and find relations to L-functions.

③ [P710, 3]) Zagier finds that the Picard-Fuchs equations associated to families of elliptic curves or families of K3 surfaces are always modular. (compositions of modular forms are solutions of the Picard-Fuchs equation) He wonders that is there any relations between families of CY varieties and automorphic varieties which relates the periods of the former to automorphic quantities.

For hyperKähler families, we have Global Torelli theorem. The image of Period map of Global Torelli can be Shimura varieties. Is this the relation that Zagier mentioned in HK cases?