Infinitesimal Torelli for Hypersurfaces
— via Griffiths Residues and Jacobian Rings —

Abstract

In this note, we present a proof of the infinitesimal Torelli theorem for smooth hypersurfaces in $\mathbb{P}^n$, following Voisin [2, Chapter 6]. The argument proceeds via Griffiths' residue calculus, the Jacobian ring, and Macaulay's duality theorem.

Contents

1 Introduction

Fix integers $n \geq 2$ and $d \geq 3$. Let $S = \mathbb{C}[X_0, \dots, X_n]$, and let $S^d \subset S$ be the subspace of homogeneous polynomials of degree d. For $f \in S^d \setminus \{0\}$, the vanishing locus $Y_f = V(f) \subset \mathbb{P}^n$ is a hypersurface of degree d. Let

$$\mathcal{U} = \{ [f] \in \mathbb{P}(S^d) \mid Y_f \text{ is smooth} \}$$

be the parameter space of all smooth degree-d hypersurfaces in $\mathbb{P}^n$. Different equations can define the same hypersurface, corresponding to a linear change of coordinates in $\mathbb{P}^n$; this redundancy is precisely the action of $\mathrm{PGL}(n+1)$ on $\mathcal{U}$ given by $g \cdot [f] = [f \circ g^{-1}]$.

Definition 1.1. The coarse moduli space of smooth embedded hypersurfaces of degree d in $\mathbb{P}^n$ is

$$\mathcal{M}_{n,d} = \mathcal{U} /\!/ \mathrm{PGL}(n+1).$$

Remark 1.2. Every smooth hypersurface of degree $d \geq 3$ in $\mathbb{P}^n$ is GIT-stable, so the GIT quotient above is well-defined. We emphasize that this moduli space parametrizes **embedded** hypersurfaces, not abstract ones. That is, it records the embedding into $\mathbb{P}^n$ up to the action of $\mathrm{PGL}(n+1)$. If we ignore this information, then, since the automorphism group of a hypersurface may not be contained in $\mathrm{PGL}(n+1)$, there may exist two different $\mathrm{PGL}(n+1)$-orbits corresponding to the same abstract hypersurface. This can only happens when

$$(n, d) = (2, 3),\ (3, 4).$$

For plane cubics this does not cause a problem: two smooth plane cubics are projectively equivalent if and only if they are isomorphic as unpointed elliptic curves. For quartic surfaces, however, this is a genuine issue: the same abstract K3 surface may carry two inequivalent degree 4 very ample polarizations, and these yield two isomorphic but not projectively equivalent quartic surfaces.

Remark 1.3. At a point $[f] \in \mathcal{U}$ with non-trivial stabiliser, some element $g \in \mathrm{PGL}(n+1)$ acts non-trivially on Y_f. When forming the quotient $\mathcal{U} \to \mathcal{M}_{n,d}$, this action identifies distinct points of Y_f, destroying the fibre structure and preventing the existence of a universal family.

The basic idea of moduli space theory is to parametrise geometric structures on manifolds of a fixed topological type by suitable invariants. For hypersurfaces the natural invariant is their primitive cohomology.

Let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^2(\mathbb{P}^n, \mathbb{Z})$ be the hyperplane class. For any smooth hypersurface Y in $\mathbb{P}^n$, we take $\eta \in H^2(Y, \mathbb{Z})$ to be the restriction of h on Y. The **primitive cohomology** of Y (with respect to η) is then defined to be the lattice

$$H^k_{\mathrm{prim}}(Y, \mathbb{Z}) = \mathrm{Ker}\big(\eta^{n-k} \cup - \colon H^k(Y, \mathbb{Z}) \to H^{2n-k}(Y, \mathbb{Z})\big)/(\mathrm{torsion}).$$

By the Lefschetz hyperplane theorem, the primitive piece $H^{n-1}_{\mathrm{prim}}(Y, \mathbb{Q})$ is the only part of the cohomology is not in the image of the restriction map from $\mathbb{P}^n$. The intersection pairing restricts to a non-degenerate bilinear form Q on $H^{n-1}_{\mathrm{prim}}(Y, \mathbb{Z})$, making it into a polarized lattice.

Remark 1.4. In general, one can define primitive cohomology for a complex projective algebraic variety Y associated to an ample class η in the same way. The ample class η is called a **polarization**. In the case of smooth hypersurfaces in $\mathbb{P}^n$, the primitive cohomology is independent of the choice of the polarization.

The variation of Hodge structures on $H^{n-1}_{\mathrm{prim}}(Y, \mathbb{Q})$ yields the period map, which we now construct.

Fix a base point $[f_0] \in \mathcal{U}$ and set $Y_0 = Y_{f_0}$, $H_{\mathrm{prim}} = H^{n-1}_{\mathrm{prim}}(Y_0, \mathbb{Z})$ with the intersection pairing Q. Let D be the Griffiths period domain parametrising all Hodge filtrations on $H_{\mathrm{prim}} \otimes \mathbb{C}$ of the given Hodge numbers that satisfy the first Riemann bilinear relation.

For each $[f] \in \mathcal{U}$, choose a small simply connected neighbourhood $V \subset \mathcal{U}$ of $[f]$. Via the Gauss–Manin connection, parallel transport along paths in V identifies $H^{n-1}_{\mathrm{prim}}(Y_t, \mathbb{Z})$ with H_{prim} for every $t \in V$; such a choice of isomorphism is a **local marking**. Transporting the Hodge filtration on $H^{n-1}_{\mathrm{prim}}(Y_t, \mathbb{C})$ to $H_{\mathrm{prim}} \otimes \mathbb{C}$ via this marking yields a point of D, and varying t defines a local holomorphic map $V \longrightarrow D$.

To obtain a global map, one must extend the marking consistently across $\mathcal{U}$. Transporting a local marking along a loop based at $[f_0]$ changes it by the monodromy action of $\pi_1(\mathcal{U}, [f_0])$. The image of this action is the **monodromy group**

$$\Gamma_{\mathcal{U}} = \mathrm{Im}\big(\pi_1(\mathcal{U}, [f_0]) \to \mathrm{Aut}(H_{\mathrm{prim}}, Q)\big) \subset \mathrm{Aut}(H_{\mathrm{prim}}, Q).$$

The local period maps therefore glue first to a holomorphic map on the universal cover $\widetilde{\mathcal{U}} \to D$, which then descends to the well-defined global map

$$\mathcal{P}_{\mathcal{U}} : \mathcal{U} \longrightarrow \Gamma_{\mathcal{U}} \backslash D.$$

Finally, a path in $\mathrm{PGL}(n+1)$ connecting $[f]$ to $g \cdot [f]$ realises the isomorphism $Y_f \cong Y_{g \cdot f}$ via monodromy; consequently $\mathcal{P}_{\mathcal{U}}$ is $\mathrm{PGL}(n+1)$-invariant and descends to the moduli space

$$\mathcal{P} : \mathcal{M}_{n,d} \longrightarrow \Gamma_{\mathcal{U}} \backslash D.$$

The Torelli problem asks to what extent a variety is determined by its Hodge structure.

Definition 1.5. We say that **global Torelli** holds for smooth degree-d hypersurfaces in $\mathbb{P}^n$ if the period map $\mathcal{P} : \mathcal{M}_{n,d} \to \Gamma_{\mathcal{U}} \backslash D$ is injective. Equivalently, two such hypersurfaces Y_f and Y_g are isomorphic if and only if there is a Hodge isometry $H^{n-1}_{\text{prim}}(Y_f, \mathbb{Z}) \xrightarrow{\sim} H^{n-1}_{\text{prim}}(Y_g, \mathbb{Z})$, i.e. an isomorphism of lattices preserving the intersection pairing and the Hodge filtration.

Global Torelli is the strongest possible statement: it asserts that the Hodge structure alone suffices to distinguish isomorphism classes. However, proving global injectivity of the period map is in general very difficult. One therefore often studies a weaker, local version first.

The tangent space of $\mathcal{M}_{n,d}$ at $[f]$ is naturally defined as the quotient

$$T_{[f]}\mathcal{M}_{n,d} = T_{[f]}\mathcal{U} \,/\, T_{[f]}(\mathrm{PGL}(n+1) \cdot [f])$$

Since the period map $\mathcal{P}_{\mathcal{U}}$ stays constant along the orbit $\mathrm{PGL}(n+1) \cdot [f]$, its derivative at $[f]$ descends to a linear map

$$d\mathcal{P}_{[f]} \colon T_{[f]}\mathcal{M}_{n,d} \longrightarrow T_{\mathcal{P}_{\mathcal{U}}([f])}(\Gamma_{\mathcal{U}} \backslash D),$$

which we call the derivative of $\mathcal{P}$ at $[f]$.

Remark 1.6. At a point $[f] \in \mathcal{U}$ with non-trivial stabiliser, the action of $\mathrm{PGL}(n+1)$ endows $\mathcal{M}_{n,d}$ with an orbifold structure at $[f]$. Thus we need to go back to $\mathcal{U}$ to define the tangent space and the tangent map at $[f]$. In practice, we usually take a transverse slice $\mathcal{S}_f \subset S^d$ to the $\mathrm{GL}(n+1)$-orbit at $[f]$. Then we identify the tangent space $T_{[f]}\mathcal{M}_{n,d}$ with $T_f\mathcal{S}_f$ and use the local period map $\mathcal{S}_f \to D$ to study the tangent map.

Definition 1.7. We say that **infinitesimal Torelli** holds at $[f] \in \mathcal{M}_{n,d}$ if the derivative $d\mathcal{P}_{[f]} : T_{[f]}\mathcal{M}_{n,d} \to T_{\mathcal{P}_{\mathcal{U}}([f])}(\Gamma_{\mathcal{U}} \backslash D)$ is injective. Equivalently, the differential of the period map separates tangent directions at $[f]$.

Infinitesimal Torelli is a much weaker condition: it only asks the period map to be an immersion, guaranteeing that small deformations of Y_f are distinguished by their Hodge structures.

In this note, we present a proof of the infinitesimal Torelli theorem for smooth hypersurfaces in $\mathbb{P}^n$. Our proof follows [2, Chapter 6].

Theorem 1.8. *Infinitesimal Torelli holds at all* $[f] \in \mathcal{M}_{n,d}$ *with* $n \geq 2$ *and* $d \geq 3$, *except* $(n, d) = (3, 3)$.

Remark 1.9. The case $(n, d) = (3, 3)$ — smooth cubic surfaces — is a true exception: all of them have isomorphic Hodge structures, but $\dim \mathcal{M}_{3,3} = 4$.

2 Hodge structure of hypersurfaces

Let X be a smooth projective variety of dimension n, and let $Y \subset X$ be a **smooth ample** divisor. Set $U = X \setminus Y$ and let $j : U \hookrightarrow X$ and $i : Y \hookrightarrow X$ denote the inclusions.

2.1 Vanishing cohomology of ample divisors

Since Y is an ample divisor in X, the Lefschetz hyperplane theorem gives

$$H^p(X, \mathbb{Z}) \xrightarrow{\simeq} H^p(Y, \mathbb{Z}) \quad (p < n-1), \qquad H^{n-1}(X, \mathbb{Z}) \hookrightarrow H^{n-1}(Y, \mathbb{Z}).$$

Thus the only part of the cohomology of Y not completely determined by the ambient space X lies in degree $n-1$. We now extract this part precisely.

Working with rational coefficients, consider the orthogonal complement, with respect to the intersection pairing on Y, of the image of the restriction:

$$H^{n-1}_{\text{van}}(Y, \mathbb{Q}) := i^* H^{n-1}(X, \mathbb{Q})^{\perp}.$$

Using the projection formula $\langle i_*\alpha, \beta\rangle_X = \langle \alpha, i^*\beta\rangle_Y$, one finds that this is precisely the kernel of the Gysin pushforward (the Poincaré dual of i^*):

$$H^{n-1}_{\mathrm{van}}(Y,\mathbb{Q}) = \mathrm{Ker}\big(i_* : H^{n-1}(Y,\mathbb{Q}) \to H^{n+1}(X,\mathbb{Q})\big).$$

It is therefore natural to consider the full Gysin exact sequence of mixed Hodge structures

$$\cdots \longrightarrow H^{p-2}(Y,\mathbb{Z})(-1) \xrightarrow{i_*} H^p(X,\mathbb{Z}) \xrightarrow{j^*} H^p(U,\mathbb{Z}) \xrightarrow{\mathrm{Res}} H^{p-1}(Y,\mathbb{Z})(-1) \longrightarrow \cdots \tag{1}$$

where (-1) is the Tate twist ($H^{p,q}(Y)(-1) \cong H^{p-1,q-1}(Y)$), and Res is the residue map.

At $p = n$, the exactness of (1) yields the following short exact sequence of mixed Hodge structures.

$$0 \longrightarrow H^n(X,\mathbb{Q}) \,/\, i_*H^{n-2}(Y,\mathbb{Q}) \longrightarrow H^n(U,\mathbb{Q}) \longrightarrow H^{n-1}_{\mathrm{van}}(Y,\mathbb{Q})(-1) \longrightarrow 0. \tag{2}$$

The Hodge filtration on $H^n(U,\mathbb{Q})$ therefore determines the Hodge structure on $H^{n-1}_{\mathrm{van}}(Y,\mathbb{Q})$.

2.2 Hodge filtration on complements

The key tool for describing the Hodge filtration on $H^n(U,\mathbb{C})$ is Deligne's logarithmic de Rham complex. See [1] for more details.

The sheaf of logarithmic 1-forms $\Omega^1_X(\log Y)$ agrees with Ω^1_X away from Y; near a point of Y, in local analytic coordinates $(z_1,\ldots,z_n)$ with $Y = \{z_1 = 0\}$, it is the locally free $\mathcal{O}_X$-module generated by

$$\frac{dz_1}{z_1},\ dz_2,\ \ldots,\ dz_n.$$

Globally, it is characterised by the exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log Y) \xrightarrow{\mathrm{Res}} \mathcal{O}_Y \longrightarrow 0,$$

where Res is the Poincaré residue along Y. The logarithmic p-forms are

$$\Omega^p_X(\log Y) = \bigwedge^p \Omega^1_X(\log Y),$$

and the exterior derivative d preserves logarithmic poles, yielding the **logarithmic de Rham complex** $(\Omega^\bullet_X(\log Y), d)$.

Since differential forms with logarithmic poles have at worst simple poles along Y, there is a natural inclusion

$$\Omega^\bullet_X(\log Y) \hookrightarrow j_*\Omega^\bullet_U$$

into the sheaf of holomorphic differential forms on U.

Theorem 2.1 ([1, Proposition 3.1.8]). *The inclusion $\Omega^\bullet_X(\log Y) \hookrightarrow j_*\Omega^\bullet_U$ is a quasi-isomorphism. Consequently,*

$$\mathbb{H}^k(X,\Omega^\bullet_X(\log Y)) \cong H^k(U,\mathbb{C}).$$

The **Hodge filtration** is obtained from the naive filtration on the logarithmic de Rham complex:

$$F^p\Omega^\bullet_X(\log Y) = \big(0 \to \Omega^p_X(\log Y) \to \Omega^{p+1}_X(\log Y) \to \cdots\big).$$

Via the quasi-isomorphism of Theorem 2.1, we set

$$F^pH^{p+q}(U,\mathbb{C}) = \mathrm{Im}\big(\mathbb{H}^{p+q}(X, F^p\Omega^\bullet_X(\log Y)) \to \mathbb{H}^{p+q}(X,\Omega^\bullet_X(\log Y)) \xrightarrow{\sim} H^{p+q}(U,\mathbb{C})\big).$$

Theorem 2.2 ([1, Corollary 3.2.13]). *The spectral sequence for the hypercohomology of the filtered complex* $(\Omega_X^\bullet(\log Y), F)$,

$$E_1^{pq} = H^q(X, \Omega_X^p(\log Y)) \implies \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log Y)) \cong H^{p+q}(U, \mathbb{C}),$$

degenerates at E_1*. In particular, the natural map*

$$\mathbb{H}^{p+q}(X, F^p\Omega_X^\bullet(\log Y)) \longrightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log Y))$$

is injective, so that

$$F^p H^{p+q}(U, \mathbb{C}) \cong \mathbb{H}^{p+q}(X, F^p\Omega_X^\bullet(\log Y)).$$

In practice, the logarithmic de Rham complex is not easy to compute directly. Deligne [1] introduced a more flexible tool: the pole-order filtration on the complex of meromorphic differential forms, which replaces logarithmic poles by controlled pole orders.

Let $j_*^m\mathcal{O}_U$ denote the sheaf of meromorphic functions on X with poles along Y. The **pole-order filtration** P on $j_*^m\mathcal{O}_U$ is the decreasing filtration defined, for $k \in \mathbb{Z}$, by

$$P^k\big(j_*^m\mathcal{O}_U\big) = \begin{cases} \mathcal{O}_X\big((1-k)Y\big), & k \le 0, \\ 0, & k > 0. \end{cases}$$

The filtration extends to the complex $j_*^m\Omega_U^\bullet$ of meromorphic differential forms by

$$P^k\big(j_*^m\Omega_U^p\big) = P^{k-p}\big(j_*^m\mathcal{O}_U\big) \otimes \Omega_X^p,$$

so that a local section of $P^k(j_*^m\Omega_U^p)$ has pole order at most $p+1-k$ along Y. The exterior derivative preserves the filtration P, and the logarithmic complex is a subcomplex of $j_*^m\Omega_U^\bullet$.

Proposition 2.3 ([1, Proposition 3.1.11]). *The natural inclusion of filtered complexes*

$$\big(\Omega_X^\bullet(\log Y),\, F\big) \;\longrightarrow\; \big(j_*^m\Omega_U^\bullet,\, P\big)$$

is a filtered quasi-isomorphism.

The proof uses local computations in a polydisk around Y; we refer to [1] for the complete proof.

2.3 Griffiths' residue theorem

We are now ready to prove Griffiths' residue theorem, which expresses the Hodge filtration on $H^n(U, \mathbb{C})$ in terms of pole orders.

Theorem 2.4 (See also [2, Theorem 6.5]). *Assume that*

$$H^i\big(X, \Omega_X^j(\ell Y)\big) = 0 \qquad \text{for all } \ell > 0,\ i > 0,\ j \ge 0. \tag{3}$$

Then for each $p \ge 1$*, the natural map*

$$\varphi_p : H^0\big(X, K_X(pY)\big) \longrightarrow H^n(U, \mathbb{C}),$$

obtained by viewing a global section of $K_X(pY)$ *as a* d*-closed meromorphic* n*-form on* U *with a pole of order* $\le p$ *along* Y*, has the following properties:*

(i) $\mathrm{Im}(\varphi_p) = F^{\,n-p+1}H^n(U, \mathbb{C})$*;*

(ii) $\mathrm{Ker}(\varphi_p)$ *consists of those* $s \in H^0(X, K_X(pY))$ *for which the corresponding* d*-closed* n*-form on* U *satisfies* $s = d\beta$*, where* β *is an* $(n-1)$*-form on* U *with pole order* $\le p-1$ *along* Y*. In particular,* φ_1 *is injective.*

Proof. By definition, φ_p is the composition

$$H^0\big(X, K_X(pY)\big) \longrightarrow \mathbb{H}^n\big(X, P^{\,n-p+1} j_*^m \Omega_U^\bullet\big) \longrightarrow \mathbb{H}^n\big(X, j_*^m \Omega_U^\bullet\big) \cong H^n(U, \mathbb{C}),$$

where the first arrow views a global section of $K_X(pY)$ as an element of the degree-n term of $P^{\,n-p+1} j_*^m \Omega_U^\bullet$, and the second is induced by the inclusion of subcomplexes.

We first analyse the first arrow. The complex $P^{\,n-p+1} j_*^m \Omega_U^\bullet$ terminates in degree n with

$$P^{\,n-p+1} j_*^m \Omega_U^n = P^{\,-p+1}\big(j_*^m \mathcal{O}_U\big) \otimes \Omega_X^n = K_X(pY).$$

The vanishing hypothesis (3) forces

$$H^i\big(X, P^{\,n-p+1} j_*^m \Omega_U^q\big) = 0 \qquad \text{for all } i > 0 \text{ and all } q,$$

so the hypercohomology spectral sequence degenerates and $\mathbb{H}^n(X, P^{\,n-p+1} j_*^m \Omega_U^\bullet)$ is the cokernel of

$$H^0\big(X, P^{\,n-p+1} j_*^m \Omega_U^{n-1}\big) \xrightarrow{d} H^0\big(X, K_X(pY)\big) \to 0.$$

Hence the first arrow is surjective, and its kernel consists of those $s \in H^0(X, K_X(pY))$ that are d of a section of $P^{\,n-p+1} j_*^m \Omega_U^{n-1} = \Omega_X^{n-1}\big((p-1)Y\big)$, i.e. meromorphic $(n-1)$-forms on U with pole order $\le p-1$.

We now analyse the second arrow. By the filtered quasi-isomorphism of Proposition 2.3,

$$\mathbb{H}^n\big(X, P^{\,n-p+1} j_*^m \Omega_U^\bullet\big) \;\cong\; \mathbb{H}^n\big(X, F^{\,n-p+1} \Omega_X^\bullet(\log Y)\big).$$

By Theorem 2.2 (taking $p + q = n$, $q = p - 1$), the natural map

$$\mathbb{H}^n\big(X, F^{\,n-p+1} \Omega_X^\bullet(\log Y)\big) \hookrightarrow \mathbb{H}^n\big(X, \Omega_X^\bullet(\log Y)\big) \cong H^n(U, \mathbb{C})$$

is injective and its image is $F^{\,n-p+1} H^n(U, \mathbb{C})$. Hence the second arrow is injective with image $F^{\,n-p+1} H^n(U, \mathbb{C})$.

Combining the two analyses, the surjectivity of the first arrow together with the injectivity of the second yields the theorem. The injectivity of φ_1 follows because $P^{\,n} j_*^m \Omega_U^{n-1} = 0$, so the first arrow and hence φ_1 is injective. □

3 Reformulation of infinitesimal Torelli via the Jacobian ring

3.1 Primitive cohomology of hypersurfaces

Recall that $S = \mathbb{C}[X_0, \dots, X_n]$ is the graded polynomial ring, with S^k its homogeneous piece of degree k. Let $f \in S^d$ be a nonsingular homogeneous polynomial of degree d, and let $Y_f \subset \mathbb{P}^n$ be the smooth hypersurface defined by f. Denote by

$$J_f = \Big(\frac{\partial f}{\partial X_0}, \dots, \frac{\partial f}{\partial X_n}\Big) \subset S$$

the **Jacobian ideal** of f, and by

$$R_f = S/J_f$$

the **Jacobian ring**. Its degree-k piece is

$$R_f^k = S^k/(J_f \cap S^k).$$

We now describe the Hodge components of $H^{n-1}_{\mathrm{prim}}(Y_f)$ in explicit algebraic terms. Recall that $h \in H^2(\mathbb{P}^n, \mathbb{Z})$ is the hyperplane class. The cohomology ring of projective space is

$$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[h]/(h^{n+1}).$$

In particular, the hypersurface Y_f has class $[Y_f] = d \cdot h$ in $H^2(\mathbb{P}^n, \mathbb{Z})$.

Consider the restriction map $i^* : H^*(\mathbb{P}^n, \mathbb{Z}) \to H^*(Y_f, \mathbb{Z})$ and the Gysin pushforward i_*. The composition $i_* \circ i^*$ is the Lefschetz operator $L = [Y_f] \cup -$. Applying it to h^{n-1} yields

$$i_*(i^* h^{n-1}) = d \cdot h^n \neq 0 \quad \text{in } H^{2n}(\mathbb{P}^n, \mathbb{Z}),$$

hence $i^* h^{n-1} \neq 0$ in $H^{2n-2}(Y_f, \mathbb{Z})$. In particular, since $H^*(\mathbb{P}^n, \mathbb{Z})$ is generated by h, this forces the restriction map $i^* : H^{n+1}(\mathbb{P}^n, \mathbb{Z}) \to H^{n+1}(Y_f, \mathbb{Z})$ to be injective. Note that $i^* \circ i_* = i^*[Y_f] \cup - = d \cdot (i^* h \cup -)$ as operators on $H^*(Y_f, \mathbb{Z})$. Taking kernels in $H^{n-1}(Y_f, \mathbb{Q})$ gives

$$H^{n-1}_{\text{van}}(Y_f, \mathbb{Q}) = \ker(i_*) = \ker(i^* \circ i_*) = \ker\big(i^* h \cup -\big) = H^{n-1}_{\text{prim}}(Y_f, \mathbb{Q}).$$

Furthermore, note that $i_* H^{n-2}(Y_f, \mathbb{Q}) = L(H^{n-2}(\mathbb{P}^n, \mathbb{Q})) = H^n(\mathbb{P}^n, \mathbb{Q})$. Setting $U_f = \mathbb{P}^n \setminus Y_f$ and substituting $X = \mathbb{P}^n$, $Y = Y_f$ into the short exact sequence (2), we obtain an isomorphism of Hodge structures

$$H^n(U_f, \mathbb{Q}) \xrightarrow{\sim} H^{n-1}_{\text{van}}(Y_f, \mathbb{Q})(-1) = H^{n-1}_{\text{prim}}(Y_f, \mathbb{Q})(-1).$$

We now bring the Jacobian ring into the picture. Fix the Euler form

$$\Omega = \sum_{i=0}^{n} (-1)^i X_i \, dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n \;\in\; H^0\big(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(n+1)\big).$$

Then for every $p \geq 1$, we have an isomorphism

$$S^{pd-n-1} \xrightarrow{\sim} H^0\big(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(pY_f)\big), \qquad P \longmapsto \frac{P\Omega}{f^p}.$$

By Bott's vanishing theorem, we have

$$H^i\big(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n}(\ell d)\big) = 0 \qquad \text{for all } \ell > 0,\ i > 0,\ j \geq 0.$$

Hence condition (3) holds for $X = \mathbb{P}^n$, $Y = Y_f$. Theorem 2.4 then yields, for each $p \geq 1$, a surjection

$$\tilde{\alpha}_p : S^{pd-n-1} \twoheadrightarrow F^{n-p} H^{n-1}_{\text{prim}}(Y_f, \mathbb{C}),\ \tilde{\alpha}_p(P) = \operatorname{Res}\Big(\frac{P\Omega}{f^p}\Big).$$

Passing to the associated graded pieces, we obtain the surjection

$$\alpha_p : S^{pd-n-1} \xrightarrow{\tilde{\alpha}_p} F^{n-p} H^{n-1}_{\text{prim}}(Y_f, \mathbb{C}) \twoheadrightarrow H^{n-p,\,p-1}_{\text{prim}}(Y_f).$$

Theorem 3.1 (See also [2, Theorem 6.10])**.** *The kernel of α_p is precisely $J_f \cap S^{pd-n-1}$; consequently α_p induces an isomorphism*

$$R_f^{pd-n-1} \xrightarrow{\sim} H^{n-p,\,p-1}_{\text{prim}}(Y_f).$$

Proof. By Theorem 2.4 we may assume $p > 1$.

We first translate the condition $P \in \operatorname{Ker}(\alpha_p)$ into a meromorphic equation. For any $P \in \operatorname{Ker}(\alpha_p)$, we have $\tilde{\alpha}_p(P) \in F^{n-p+1} H^{n-1}_{\text{prim}}(Y_f)$. By Theorem 2.4 the map $\tilde{\alpha}_{p-1} \colon S^{(p-1)d-n-1} \to F^{n-p+1} H^{n-1}_{\text{prim}}(Y_f)$ is surjective. Hence there exists $Q \in S^{(p-1)d-n-1}$ such that $\tilde{\alpha}_p(P) = \tilde{\alpha}_{p-1}(Q)$, or equivalently, $\tilde{\alpha}_p(P - fQ) = 0$. By Theorem 2.4(ii), there exists $\gamma \in H^0\big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y_f)\big)$ such that

$$\frac{(P - fQ)\,\Omega}{f^p} = d\gamma,$$

or equivalently,

$$\frac{P\Omega}{f^p} \equiv d\gamma \pmod{H^0\big(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((p-1)Y_f)\big)}. \tag{4}$$

Conversely, if (4) holds, then $P\Omega/f^{\,p} - d\gamma = \beta$ where β has pole $\le p-1$ along Y_f. Writing $\beta = Q\Omega/f^{\,p-1}$ with $Q \in S^{(p-1)d-n-1}$, we obtain $\tilde{\alpha}_p(P) = \tilde{\alpha}_{p-1}(Q)$, which implies that $P \in \operatorname{Ker}(\alpha_p)$. In summary,

$$P \in \operatorname{Ker}(\alpha_p) \iff \exists\, \gamma \in H^0\big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y_f)\big) \text{ such that } \frac{P\Omega}{f^{\,p}} \equiv d\gamma \pmod{H^0\big(\mathbb{P}^n, \Omega^{n}_{\mathbb{P}^n}((p-1)Y_f)\big)}. \tag{5}$$

Next we parametrise the $(n-1)$-form γ. Write $\partial_i = \frac{\partial}{\partial X_i}$ and let $\lrcorner$ denote the contraction. By the Euler exact sequence on $\mathbb{P}^n$ we have

$$T_{\mathbb{P}^n} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(1)\cdot \partial_i / \mathcal{O}_{\mathbb{P}^n} \cdot \sum_{i=0}^{n} X_i \partial_i$$

Furthermore, contraction with the Euler form Ω defines an isomorphism of sheaves

$$T_{\mathbb{P}^n} \xrightarrow{\sim} \Omega^{n-1}_{\mathbb{P}^n}(n+1), \qquad v \longmapsto v \lrcorner \Omega.$$

Since $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((p-1)d-n-1)) = 0$, tensoring with $\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{P}^n}((p-1)Y_f)$ and taking global sections, we obtain a surjection

$$\bigoplus_{i=0}^{n} S^{(p-1)d-n} \cdot \partial_i \lrcorner \Omega \twoheadrightarrow H^0\big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y_f)\big).$$

Concretely, every $\gamma \in H^0\big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)Y_f)\big)$ can be lifted to $\mathbb{C}^{n+1} \setminus \{0\}$ and written as

$$\gamma = \frac{1}{f^{\,p-1}} \sum_{i=0}^{n} P_i\, (\partial_i \lrcorner \Omega) \qquad \text{for some } P_i \in S^{(p-1)d-n}. \tag{6}$$

Remark 3.2. The representation (6) is not unique. Any choice of the representative works for our computation. The following computation should be interpreted as happening on the affine space $\mathbb{C}^{n+1} \setminus \{f = 0\}$. The representation (6) itself ensures that the computation descends to $\mathbb{P}^n \setminus Y_f$.

Now we compute $d\gamma$ explicitly. Denote the auxiliary $(n-1)$-forms

$$\Xi_i = \partial_i \lrcorner (dX_0 \wedge \cdots \wedge dX_n) = (-1)^i\, dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n,$$

and recall the contraction identity for any vector field v and 1-form α:

$$\alpha \wedge (v \lrcorner \Omega) = -\, v \lrcorner (\alpha \wedge \Omega) + \langle \alpha, v \rangle\, \Omega.$$

Then we obtain the following equalities:

$$
\begin{aligned}
d\gamma &= \frac{1}{f^{\,p-1}} \cdot d\Big(\sum_{i=0}^{n} P_i \,(\partial_i \lrcorner \Omega)\Big) - (p-1)\,\frac{df}{f^{\,p}} \wedge \sum_{i=0}^{n} P_i \,(\partial_i \lrcorner \Omega) \\
&\qquad \text{(by the Leibniz rule)} \\
&= \frac{1}{f^{\,p-1}} \cdot d\Big(\sum_{i=0}^{n} P_i \,(\partial_i \lrcorner \Omega)\Big) - \frac{p-1}{f^{\,p}} \sum_{i,j=0}^{n} P_i \,\partial_j f \; dX_j \wedge (\partial_i \lrcorner \Omega) \\
&= \frac{1}{f^{\,p-1}} \cdot d\Big(\sum_{i=0}^{n} P_i \,(\partial_i \lrcorner \Omega)\Big) - \frac{p-1}{f^{\,p}} \sum_{i,j=0}^{n} P_i \,\partial_j f \,\big(-\,\partial_i \lrcorner (dX_j \wedge \Omega) + \delta_{ij}\,\Omega\big) \\
&\qquad \text{(contraction identity with } \alpha = dX_j,\ v = \partial_i\text{)} \\
&= \frac{1}{f^{\,p-1}} \cdot d\Big(\sum_{i=0}^{n} P_i \,(\partial_i \lrcorner \Omega)\Big) - \frac{p-1}{f^{\,p}} \sum_{i,j=0}^{n} P_i \,\partial_j f \,\big(-\,X_j\,\Xi_i + \delta_{ij}\,\Omega\big) \\
&= \frac{1}{f^{\,p-1}} \Big(d\Big(\sum_{i=0}^{n} P_i \,(\partial_i \lrcorner \Omega)\Big) + (p-1)\cdot d \cdot \sum_{i=0}^{n} P_i \,\Xi_i \Big) - \frac{p-1}{f^{\,p}} \sum_{i=0}^{n} P_i \,\partial_i f \,\Omega.
\end{aligned}
$$

The last equality holds by the Euler relation $\sum_j X_j \partial_j f = d \cdot f$. Modulo those n-forms with pole order $\leq$ along Y_f, we have

$$
d\gamma \equiv -\frac{p-1}{f^{\,p}} \sum_{i=0}^{n} P_i \,\partial_i f \,\Omega \quad \big(\mathrm{mod}\ H^0\big(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((p-1)Y_f)\big)\big). \tag{7}
$$

Finally, inserting the parametrisation (6) into the equivalence (5) and using the computation (7), we obtain

$$
P \in \mathrm{Ker}(\alpha_p) \iff \exists\, P_i \in S^{(p-1)d-n} \text{ such that } P \equiv -(p-1) \sum_{i=0}^{n} P_i \,\partial_i f \quad (\mathrm{mod}\ f \cdot S^{(p-1)d-n-1}).
$$

Since $\sum_j X_j \partial_j f = d \cdot f$, the right-hand side means precisely $P \in J_f \cap S^{pd-n-1}$. Thus $\mathrm{Ker}(\alpha_p) = J_f \cap S^{pd-n-1}$, completing the proof of Theorem 3.1. □

3.2 The infinitesimal period map

We now compute the tangent space $T_{[f]}\mathcal{M}_{n,d}$ in algebraic terms. By definition, we have

$$
T_{[f]}\mathcal{M}_{n,d} = T_{[f]}\mathcal{U} \,/\, T_{[f]}(\mathrm{PGL}(n+1)\cdot[f]) = S^d \,/\, T_f(\mathrm{GL}(n+1)\cdot f)
$$

The group $\mathrm{GL}(n+1)$ acts on f by $g \cdot f = f \circ g^{-1}$ for $g \in \mathrm{GL}(n+1)$. Differentiating this action at the identity yields the infinitesimal action

$$
\mathfrak{gl}(n+1) \longrightarrow S^d, \qquad A = (a_{ij})_{0 \leq i,j \leq n} \longmapsto -\sum_{i,j=0}^{n} a_{ij}\, X_j \,\frac{\partial f}{\partial X_i}.
$$

Its image is the subspace of S^d spanned by $\{X_j \partial_i f\}_{i,j=0}^{n}$, which is exactly $J_f \cap S^d$. Hence we obtain

$$
T_{[f]}\mathcal{M}_{n,d} = S^d \,/\, (J_f \cap S^d) = R_f^d. \tag{8}
$$

Remark 3.3. Take a class $[H] \in R_f^d$ and choose a representative $H \in S^d$. Then the one-parameter family

$$\{[f + \varepsilon H] \in \mathcal{U} \mid |\varepsilon| \ll 1\} \subset \mathcal{U}$$

defines a first-order infinitesimal deformation of Y_f. Consequently, R_f^d parametrizes first-order embedded deformations of the smooth hypersurface Y_f modulo projective equivalence.

Remark 3.4. R_f^d is exactly the image of the Kodaira-Spencer map $H^0(Y_f, O_{Y_f}(d)) \to H^1(Y_f, T_{Y_f})$.

Having identified the tangent space, we now compute the derivative of the period map at $[f]$

$$d\mathcal{P}_{[f]} : T_{[f]}\mathcal{M}_{n,d} \longrightarrow T_{\mathcal{P}_{\mathcal{U}}([f])}(\Gamma_{\mathcal{U}}\backslash D).$$

By Griffiths transversality (see [2, §5.3]), the image of $d\mathcal{P}_f$ is contained in the horizontal tangent subspace $\bigoplus_p \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p) \subset T_{\mathcal{P}(f)}D$. The derivative therefore decomposes as

$$d\mathcal{P}_f = \bigoplus_{p=1}^{n} d\mathcal{P}_f^{n-p}, \qquad d\mathcal{P}_f^{n-p} : T_{[f]}\mathcal{M}_{n,d} \longrightarrow \mathrm{Hom}\big(H^{n-p,\,p-1}_{\mathrm{prim}}(Y_f),\, H^{n-p-1,\,p}_{\mathrm{prim}}(Y_f)\big). \tag{9}$$

Substituting the isomorphisms

$$T_{[f]}\mathcal{M}_{n,d} \;\cong\; R_f^d, \qquad H^{n-p,\,p-1}_{\mathrm{prim}}(Y_f) \;\cong\; R_f^{pd-n-1}$$

into (9), each component becomes a map between graded pieces of the Jacobian ring:

$$d\mathcal{P}_f^{n-p} : R_f^d \longrightarrow \mathrm{Hom}\big(R_f^{pd-n-1},\, R_f^{(p+1)d-n-1}\big).$$

It remains to determine this map explicitly. The computation yields the following result.

Theorem 3.5 (See also [2, Theorem 6.13])**.** *Under the Griffiths residue identification, the derivative is given by multiplication in the Jacobian ring:*

$$d\mathcal{P}_f^{n-p}([H])([P]) = -p \cdot [HP] \quad \textit{in } R_f^{(p+1)d-n-1},$$

for $[H] \in R_f^d$ *and* $[P] \in R_f^{pd-n-1}$.

Proof. A class $[P] \in R_f^{pd-n-1}$ corresponds to the residue

$$\left[\mathrm{Res}\left(\frac{P\Omega}{f^p}\right)\right] \in H^{n-p,\,p-1}_{\mathrm{prim}}(Y_f),$$

which can be extended to a period along the one-parameter family $\{[f + \varepsilon H] \in \mathcal{U} \mid |\varepsilon| \ll 1\}$:

$$\epsilon \mapsto \left[\mathrm{Res}\left(\frac{P\Omega}{(f+\varepsilon H)^p}\right)\right].$$

Remark 3.6. Take a tubular neighborhood T of Y_f such that $Y_{f+\epsilon H} \subset T$ for all sufficiently small ϵ. Then under the identification

$$H^n(\mathbb{P}^n \setminus T, \mathbb{C}) \simeq H^n(U_{f+\epsilon H}, \mathbb{C}) \simeq H^{n-1}_{\mathrm{prim}}(Y_{f+\epsilon H}, \mathbb{C}),$$

the period above is exactly

$$\epsilon \mapsto \left[\frac{P\Omega}{(f+\varepsilon H)^p}\right] \in H^n(\mathbb{P}^n \setminus T, \mathbb{C}),$$

which is obviously smooth.

By definition, the tangent map acts by differentiating this period

$$d\mathcal{P}_f^{n-p}([H])([P]) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[\mathrm{Res}\left(\frac{P\Omega}{(f+\varepsilon H)^p}\right)\right] = -p\,\left[\mathrm{Res}\left(\frac{PH\Omega}{f^{p+1}}\right)\right].$$

This last residue corresponds to $-p \cdot [HP]$ in $R_f^{(p+1)d-n-1}$, which establishes the formula. □

4 Proof of the main theorem

4.1 The Macaulay duality theorem

We now state the algebraic result that guarantees the injectivity of the multiplication maps. The central algebraic notion is the following.

Definition 4.1. A sequence $G_0, \dots, G_n \in S$ of homogeneous polynomials is a **regular sequence** if G_i is a non-zero-divisor in $S/(G_0, \dots, G_{i-1})$ for each i. For a homogeneous regular sequence, this is equivalent to the condition that the G_i have no common zero in $\mathbb{P}^n$.

Macaulay's theorem describes the structure of the quotient $R_{G.} = S/(G_0, \dots, G_n)$ for a regular sequence.

Theorem 4.2 (See also [2, Theorem 6.19])**.** *Let $G_0, \dots, G_n \in S = \mathbb{C}[X_0, \dots, X_n]$ be a regular sequence of homogeneous polynomials of degrees $d_0, \dots, d_n$, and let $R_{G.} = S/(G_0, \dots, G_n)$. Set $N = \sum d_i - n - 1$. Then $R^k_{G.} = 0$ for $k > N$, $\dim_{\mathbb{C}} R^N_{G.} = 1$, and the multiplication pairing*

$$R^k_{G.} \times R^{N-k}_{G.} \longrightarrow R^N_{G.} \cong \mathbb{C}$$

induced by multiplication is perfect.

The proof of Theorem 4.2 rests on the following elementary fact from linear algebra.

Lemma 4.3. *Let V be an $(n+1)$-dimensional $\mathbb{C}$-vector space and $\sigma \in V^* \setminus \{0\}$ a non-zero linear form. Then the Koszul complex*

$$0 \longrightarrow \bigwedge^{n+1} V \xrightarrow{\sigma_{n+1}} \bigwedge^{n} V \xrightarrow{\sigma_n} \cdots \xrightarrow{\sigma_2} V \xrightarrow{\sigma_1 = \sigma} \mathbb{C} \longrightarrow 0,$$

where $\sigma_p(v_1 \wedge \cdots \wedge v_p) = \sum_{j=1}^p (-1)^{j-1} \sigma(v_j)\, v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_p$, is exact.

Proof. Choose a basis $e_0, \dots, e_n$ of V such that $\sigma(e_0) = 1$ and $\sigma(e_i) = 0$ for $i > 0$. Then σ_p sends a wedge product containing e_0 to the complementary wedge product (with a sign) and annihilates products not containing e_0. A direct verification shows $\ker \sigma_p = \operatorname{Im} \sigma_{p+1}$ for all p, yielding exactness. □

We now turn to the proof of Macaulay's theorem. The argument proceeds by constructing a sheaf-theoretic Koszul resolution and computing its hypercohomology via a spherical spectral sequence.

Proof of Theorem 4.2. Consider the vector bundle $E = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(-d_i)$ of rank $n+1$ on $\mathbb{P}^n$. Each G_i defines a morphism $\mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{O}_{\mathbb{P}^n}$ by multiplication, and summing them gives

$$G. = (G_0, \dots, G_n) : E \longrightarrow \mathcal{O}_{\mathbb{P}^n}, \qquad (s_0, \dots, s_n) \longmapsto \sum_{i=0}^n G_i s_i.$$

The key observation is that

$$R^k_{G.} = \operatorname{coker}\bigl(H^0(G.(k)) : H^0(E(k)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(k))\bigr).$$

Our aim is to relate this cokernel, via the Koszul complex of sheaves and its hypercohomology, to a kernel on the Serre-dual side, thereby obtaining the perfect pairing.

To this end, we first form the Koszul complex. The morphism $G.$ may be viewed as a global section of E^*; by contraction it induces maps $\delta_p : \bigwedge^p E \to \bigwedge^{p-1} E$ for each p. We obtain the complex of locally free sheaves

$$K^\bullet : \qquad 0 \longrightarrow \bigwedge^{n+1} E \xrightarrow{\delta_{n+1}} \bigwedge^{n} E \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} E \xrightarrow{\delta_1 = G.} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0. \tag{10}$$

Exactness is checked pointwise: for every $x \in \mathbb{P}^n$, the fibre $E_x \cong \mathbb{C}^{n+1}$ and $G.(x) \in E_x^*$ is non-zero. Lemma 4.3 applied to the vector space E_x and the linear form $G.(x)$ shows that the fibre of $K^\bullet$ at x is exact; hence $K^\bullet$ is an exact complex of sheaves.

For any integer k, twisting (10) by $\mathcal{O}_{\mathbb{P}^n}(k)$ yields an exact complex $K^\bullet(k)$. So we have

$$E_1^{p,q} = H^q\big(\mathbb{P}^n, \bigwedge^p E(k)\big) \implies \mathbb{H}^{p+q}\big(\mathbb{P}^n, K^\bullet(k)\big) = 0.$$

Since each $\bigwedge^p E(k)$ is a direct sum of line bundles, Bott's vanishing theorem gives

$$H^q\big(\mathbb{P}^n, \bigwedge^p E(k)\big) = 0 \qquad (q \neq 0, n).$$

Thus the spectral sequence has non-zero entries only in rows $q = 0$ and $q = n$ (it is a *spherical* spectral sequence). Since the spectral sequence converges to zero, on the E_2 page only $E_2^{0,n}$ and $E_2^{n+1,0}$ can be non-zero, and they must cancel through d_{n+1}, yielding an isomorphism

$$d_{n+1} : E_2^{0,n} \xrightarrow{\sim} E_2^{n+1,0}.$$

We now compute these two E_2 terms. By definition,

$$E_2^{n+1,0} = H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \big/ G.H^0(E(k)) = R_{G.}^k.$$

and

$$E_2^{0,n} = \mathrm{Ker}\big(d_1 = (\delta_{n+1})_* : H^n(\mathbb{P}^n, \bigwedge^{n+1} E(k)) \to H^n(\mathbb{P}^n, \bigwedge^n E(k))\big).$$

Note that $\bigwedge^{n+1} E = \mathcal{O}_{\mathbb{P}^n}(-\sum d_i)$, and the perfect pairing $\bigwedge^n E \otimes E \to \bigwedge^{n+1} E$ induces that $\bigwedge^n E \cong E^*\big(-\sum d_i\big)$. So we have

$$\bigwedge^{n+1} E(k) = \mathcal{O}_{\mathbb{P}^n}(k - \sum d_i), \qquad \bigwedge^n E(k) \cong E^*\big(k - \sum d_i\big).$$

By Serre duality on $\mathbb{P}^n$, the map

$$(\delta_{n+1})_* : H^n\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k - \sum d_i)\big) \longrightarrow H^n\big(\mathbb{P}^n, E^*(k - \sum d_i)\big)$$

is dual to the map on global sections

$$G. : H^0\big(\mathbb{P}^n, E(\sum d_i - k - n - 1)\big) \longrightarrow H^0\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\sum d_i - k - n - 1)\big),$$

which implies that

$$E_2^{0,n} \cong \big(R_{G.}^{\sum d_i - k - n - 1}\big)^* = \big(R_{G.}^{N-k}\big)^*.$$

The isomorphism $d_{n+1} : E_2^{0,n} \xrightarrow{\sim} E_2^{n+1,0}$ from above therefore yields an isomorphism

$$\big(R_{G.}^{N-k}\big)^* \xrightarrow{\sim} R_{G.}^k,$$

which induces, for each k, a perfect pairing

$$\langle -, - \rangle : R_{G.}^{N-k} \times R_{G.}^k \longrightarrow R_{G.}^N \cong \mathbb{C}. \tag{11}$$

In particular, $R_{G.}^k = 0$ for $k > N$, and setting $k = N$ gives an isomorphism

$$R_{G.}^N \xrightarrow{\sim} \mathbb{C}, \qquad A \mapsto \langle 1, A \rangle, \tag{12}$$

where $1 \in R_{G.}^0$ is the unit.

It remains to verify that (11) yields the desired perfect pairing. For $P \in S^\ell$, multiplication by the homogeneous polynomial P induces a morphism of complexes $K^\bullet(k) \to K^\bullet(k+\ell)$ (acting as multiplication on each term $\mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_{\mathbb{P}^n}(m+\ell)$). This morphism is compatible with the spectral sequence and therefore commutes with the differential d_{n+1}. On the E_2 page it induces, for the cokernel term, the multiplication map $P \cdot (-) : R^k_{G.} \to R^{k+\ell}_{G.}$, and for the kernel term its dual P^*. The commutativity with d_{n+1} thus translates into the commutative diagram

$$\begin{array}{ccc} (R^{N-k}_{G.})^* & \xrightarrow{\sim} & R^k_{G.} \\ {\scriptstyle P^*}\downarrow & & \downarrow{\scriptstyle P\cdot(-)} \\ (R^{N-k-\ell}_{G.})^* & \xrightarrow{\sim} & R^{k+\ell}_{G.} \end{array}$$

where the horizontal isomorphisms are those of (11). The commutativity of this diagram implies the reduction identity

$$\langle P, Q\rangle = \langle 1, PQ\rangle \qquad (\forall\, P \in R^{N-k}_{G.},\, Q \in R^k_{G.}).$$

Thus, under the identification $R^N_{G.} \cong \mathbb{C}$ given in (12), the pairing $\langle P, Q\rangle$ corresponds precisely to the product $PQ \in R^N_{G.}$. In particular the pairing is perfect. This completes the proof. □

Corollary 4.4. *Let $R_{G.}$ be as in Theorem 4.2.*

(i) $R^k_{G.} \neq 0$ if and only if $0 \le k \le N$.

(ii) For integers a, b with $b \ge 0$ and $a + b \le N$, the multiplication map

$$\mu : R^a_{G.} \longrightarrow \mathrm{Hom}\big(R^b_{G.},\, R^{a+b}_{G.}\big), \qquad A \mapsto (B \mapsto AB),$$

is injective.

Proof. (i) is immediate from the perfect pairing: if $k > N$ then $N - k < 0$, so $R^{N-k}_{G.} = 0$, hence $R^k_{G.} = 0$ by duality. Conversely, if $R^k_{G.} = 0$ then $R^{N-k}_{G.} = 0$ as well, forcing $k \notin [0, N]$.

For (ii), let $A \in \ker \mu$. Then $AB = 0$ in $R^{a+b}_{G.}$ for every $B \in R^b_{G.}$. For any $\varphi \in R^{N-a-b}_{G.}$ we have $AB\varphi = 0$ in $R^N_{G.}$. Since $b \ge 0$ and $N - a - b \ge 0$, the multiplication map $R^{N-a-b}_{G.} \otimes R^b_{G.} \to R^{N-a}_{G.}$ is surjective by the perfectness of the pairing. Hence $AC = 0$ in $R^N_{G.}$ for all $C \in R^{N-a}_{G.}$, and non-degeneracy of the pairing forces $A = 0$. □

4.2 Completion of the proof of Theorem 1.8

For the Jacobian ring R_f, the partial derivatives $\partial_i f$ form a regular sequence of common degree $d-1$. Macaulay's theorem applies with $d_i = d - 1$ for all $i = 0, \dots, n$, giving

$$N = (n+1)(d-1) - (n+1) = (n+1)(d-2).$$

Recall from §3.2 that $d\mathcal{P}_f = \bigoplus_{p=1}^n d\mathcal{P}_f^{n-p}$, and each component is (up to the non-zero scalar $-p$) the multiplication map

$$\mu_p : R^d_f \longrightarrow \mathrm{Hom}\big(R^{pd-n-1}_f,\, R^{(p+1)d-n-1}_f\big), \qquad H \mapsto (P \mapsto HP). \tag{13}$$

Since $d\mathcal{P}_f$ is a direct sum, it is injective as soon as a single component μ_p is injective. By Corollary 4.4(ii), for a given p the map μ_p is injective if

$$0 \le pd - n - 1 \le N - d. \tag{14}$$

Thus it suffices to exhibit one p_0 fulfilling this numerical condition. We take

$$p_0 = \lceil \tfrac{n+1}{d} \rceil.$$

Then the numerical condition (14) reduces to

$$(n - p_0)d - n - 1 \geq 0$$

Using the standard estimate $p_0 \leq \frac{n+d}{d}$, it is enough to have

$$(n - 1)(d - 2) \geq 3.$$

For $(n - 1)(d - 2) < 3$ with $n \geq 2$, $d \geq 3$, and $(n, d) \neq (3, 3)$, only two pairs remain:

- $(n, d) = (2, 3)$: $p_0 = \lceil 3/3 \rceil = 1$, the condition $(n - p_0)d - n - 1 = 0$ holds.
- $(n, d) = (2, 4)$: $p_0 = \lceil 3/4 \rceil = 1$, the condition $(n - p_0)d - n - 1 = 1 > 0$ holds.

Thus for every $(n, d) \neq (3, 3)$ with $n \geq 2$, $d \geq 3$, the numerical condition (14) is satisfied. This completes the proof of Theorem 1.8.

References

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