6

Hodge Filtration of Hypersurfaces

This chapter, which is devoted to the variations of Hodge structure of complete families of sufficiently ample hypersurfaces, reveals the algebraic nature of the theory of infinitesimal variations of Hodge structure. We give an explicit description of the complexes $\mathcal{K}_{p,q}$ introduced above, at least in the case of hypersurfaces of projective space. (One can find a similar description in Green (1984c) for the sufficiently ample hypersurfaces of any smooth variety.)

The first result, and maybe the most important one conceptually speaking, is the realisation of the vanishing cohomology of a hypersurface $Y \subset X$ as above, defined by a homogeneous equation $f = 0$, using residues $\mathrm{Res}_Y \eta/f^p$ for $\eta \in H^0(X, K_X(pY))$ of meromorphic forms with poles along Y, together with the fact that this realisation relates the Hodge level and the order p of the pole, as in the following theorem.

Theorem 6.1 (Griffiths) *The residues* $\mathrm{Res}_Y \eta/f^p$ *for* $\eta \in H^0(X, K_X(pY))$ *generate* $F^{n-p}H^{n-1}(Y)_{\mathrm{van}}$, *where* $n = \dim X$.

The second result, more computational in nature, allows us to deduce the following description of the Dolbeault cohomology of a hypersurface $Y \subset \mathbb{P}^n$ of degree d from the preceding theorem. Letting S denote the ring of homogeneous polynomials on $\mathbb{P}^n$ and Ω a generator of $H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(n+1))$, we have the following result.

Proposition 6.2 *The kernel of the map* $\overline{\alpha}_p : S^{pd-n-1} \to \mathrm{Gr}_F^{n-p} H^{n-1}(Y)_{\mathrm{van}}$, *which to* P *associates*

$$\overline{\alpha}_p(P) = \mathrm{Res}_Y P\Omega/f^p \mod F^{n-p+1}H^{n-1}(Y)_{\mathrm{van}},$$

https://doi.org/10.1017/CBO9780511615177.007 Published online by Cambridge University Press

is equal to the degree $pd - n - 1$ *component* J_f^{pd-n-1} *of the Jacobian ideal of* f*, and thus induces an isomorphism* $R_f^{pd-n-1} \cong H^{n-p,p-1}(Y)_{\mathrm{van}}$*, where* $R_f = S/J_f$ *is the Jacobian ring of* f*.*

The last important result we introduce here concerns the computation of the map

$$\overline{\nabla} : H^{n-p,p-1}(Y)_{\mathrm{van}} \to \mathrm{Hom}\,(S^d, H^{n-p-1,p}(Y)_{\mathrm{van}}),$$

which describes the infinitesimal variation of Hodge structure on the vanishing (or primitive) cohomology of the hypersurfaces Y_t at the point f. Here, S^d is viewed as the tangent space of the universal family of hypersurfaces at the point f. One can show that via the preceding identification, this arrow can be identified up to a coefficient with the map given by multiplication

$$R_f^{pd-n-1} \to \mathrm{Hom}\left(S^d, R_f^{(p+1)d-n-1}\right).$$

The remainder of the chapter is devoted to the study of the algebraic properties of the Jacobian rings of hypersurfaces of projective space. On the one hand, we prove Macaulay's theorem, which in modern language states that these rings are Gorenstein rings, and on the other hand, we prove the symmetriser lemma. From the point of view of infinitesimal variations of Hodge structure, these statements imply the exactness of the complexes $\mathcal{K}_{p,q}$, $p + q = n - 1$, in degree 0 for $q < n - 1$ and in degree 1 for $q < n - 2$, whenever d is sufficiently large. We can deduce a generalisation of the Noether–Lefschetz theorem in the following form. Let $U \subset S^d$ be an open set parametrising smooth hypersurfaces, and let λ be a non-zero section of the local system $H^{n-1}_{\mathbb{C},\mathrm{ev}}$ on U.

Theorem 6.3 *If the integer d is sufficiently large, then the Hodge locus*

$$U_\lambda^1 = \{t \in U \mid \lambda_t \in F^1 H^{n-1}(Y_t)\}$$

is a proper analytic subset of U*.*

In chapter 7, we will give an application of the symmetriser lemma to the study of the Abel–Jacobi map.

6.1 Filtration by the order of the pole

6.1.1 Logarithmic complexes

Let us recall some results from vI.8.2 and vI.8.4. Let X be a projective variety, and $Y \overset{l}{\hookrightarrow} X$ a smooth hypersurface. Set

$$U = X - Y \overset{j}{\hookrightarrow} X.$$

Recall (see vI.8.2.2) that the logarithmic de Rham complex $(\Omega_X^\bullet(\log Y), d)$ is the complex of free $\mathcal{O}_X$-modules defined as follows:

$$\Omega_X^k(\log Y) = \bigwedge^k \Omega_X(\log Y),$$

where $\Omega_X(\log Y)$ is the sheaf of free $\mathcal{O}_X$-modules locally generated by Ω_X and df/f, where f is a local holomorphic equation for Y. The differential is the exterior differential. This complex can be viewed as a subcomplex of the complex $j_*\mathcal{A}_U^\bullet$, and we have the following result, which was proved in vI.8.2.3.

Theorem 6.4 *The inclusion*

$$\Omega_X^\bullet(\log Y) \hookrightarrow j_*\mathcal{A}_U^\bullet$$

is a quasi-isomorphism.

As the de Rham complex $(\mathcal{A}_U^\bullet, d)$ is a resolution of the constant sheaf $\mathbb{C}$ on U, this implies that

$$H^k(U, \mathbb{C}) \cong \mathbb{H}^k(X, j_*\mathcal{A}_U^\bullet) \cong \mathbb{H}^k(X, \Omega_X^\bullet(\log Y)),$$

where the first equality follows from the fact that each $\mathcal{A}_U^k$ is fine and thus acyclic for the functor j_*. We then define the Hodge filtration $F^\cdot$ on each $H^k(U, \mathbb{C})$ as the filtration induced by the Hodge filtration $F^\cdot$ on the logarithmic de Rham complex

$$F^p\Omega_X^\bullet(\log Y) = 0 \to \Omega_X^p(\log Y) \to \Omega_X^{p+1}(\log Y) \to \cdots.$$

Furthermore, we have two morphisms of complexes: the inclusion

$$\Omega_X^\bullet \hookrightarrow \Omega_X^\bullet(\log Y),$$

which in cohomology induces the restriction

$$H^k(X, \mathbb{C}) \to H^k(U, \mathbb{C}), \tag{6.1}$$

and the residue

$$\mathrm{Res} : \Omega_X^\bullet(\log Y) \to \Omega_Y^{\bullet-1},$$

which to $\alpha \wedge (df/f)$ associates $\mathrm{Res}\,\alpha \wedge (df/f) = 2i\pi\alpha_{|Y}$. The map induced by Res in cohomology is the residue

$$H^k(U, \mathbb{C}) \to H^{k-1}(Y, \mathbb{C}), \tag{6.2}$$

which we can also define as the composition

$$H^k(U, \mathbb{C}) \to H^{k+1}(X, U, \mathbb{C}) \cong H^{k+1}(T, \partial T, \mathbb{C}) \cong H^{k-1}(Y, \mathbb{C}),$$

where T is a tubular neighbourhood of Y in X, the first arrow is the connecting morphism in the long exact sequence of relative cohomology, the second is the excision isomorphism, and the last is the Thom isomorphism.

In particular, we see that the morphisms (6.1) and (6.2) are defined over $\mathbb{Z}$ and compatible with the Hodge filtrations. (More precisely, the morphism Res sends $F^pH^k(U, \mathbb{C})$ to $F^{p-1}H^{k-1}(Y, \mathbb{C})$, so that it is compatible with the Hodge filtrations after shifting the bidegrees of the Hodge structure on $H^{k-1}(Y, \mathbb{C})$ by $(1, 1)$: we call this operation a Tate twist. It also makes it possible to ingeniously get rid of the coefficient $2\pi i$ in the definition of the residue.)

Note that by the description given above, the residue is part of the long exact sequence of relative cohomology of the pair (X, U), which thanks to the Thom isomorphism $H^{k+1}(X, U, \mathbb{Z}) \cong H^{k-1}(Y, \mathbb{Z})$ given above, can be written

$$\cdots H^k(X, \mathbb{Z}) \to H^k(U, \mathbb{Z}) \xrightarrow{\mathrm{Res}} H^{k-1}(Y, \mathbb{Z}) \to H^{k+1}(X, \mathbb{Z}) \cdots.$$

One can show (see vI.8.4.2) that the last arrow is the Gysin morphism

$$l_* : H^{k-1}(Y, \mathbb{Z}) \to H^{k+1}(X, \mathbb{Z}).$$

Now, assume that $\dim X = n = k$ and that Y is an ample hypersurface of X. Then, passing to the rational cohomology, by the Lefschetz decomposition associated to the polarisation given by $\mathcal{O}_X(Y)$, we have

$$H^n(X, \mathbb{Q})/l_*H^{k-2}(Y, \mathbb{Q}) = H^n(X, \mathbb{Q})_{\mathrm{prim}},$$
$$\mathrm{Ker}\,(l_* : H^{n-1}(Y, \mathbb{Q}) \to H^{n+1}(X, \mathbb{Q})) =: H^{n-1}(Y, \mathbb{Q})_{\mathrm{van}}$$

and a short exact sequence compatible with the Hodge filtrations:

$$0 \to H^n(X, \mathbb{Q})_{\mathrm{prim}} \xrightarrow{j^*} H^n(U, \mathbb{Q}) \xrightarrow{\mathrm{Res}} H^{n-1}(Y, \mathbb{Q})_{\mathrm{van}} \to 0. \tag{6.3}$$

https://doi.org/10.1017/CBO9780511615177.007 Published online by Cambridge University Press

The morphisms j^* and Res are in fact strictly compatible with the Hodge filtrations by theorem 4.20, since the weight filtration W

$$W_0 H^n(U, \mathbb{Q}) = j^* H^n(X, \mathbb{Q})_{\text{prim}}, \quad W_1 H^n(U, \mathbb{Q}) = H^n(U, \mathbb{Q})$$

and the Hodge filtration equip $H^n(U, \mathbb{Q})$ with a mixed Hodge structure (after shifting the bidegree of the Hodge structure on $H^{n-1}(Y, \mathbb{Q})_{\text{van}}$ as above).

6.1.2 Hodge filtration and filtration by the order of the pole

Let us now make the following hypotheses:

(*) *For every* $k > 0,\ i > 0,\ j \geq 0$, *we have*

$$H^i\big(X, \Omega_X^j(kY)\big) = 0,$$

where $\Omega_X^j(kY) = \Omega_X^j \otimes \mathcal{O}_X(Y)^{\otimes k}$.

These hypotheses are satisfied by Serre's vanishing theorem (see volume I, chapter 7, exercise 2) if Y is sufficiently ample in X, i.e. if $\mathcal{O}_X(Y)$ is a sufficiently large multiple of an ample line bundle on X. They are also satisfied by Bott's vanishing theorem if X is the projective space $\mathbb{P}^n$.

Under these hypotheses (which can actually be weakened, as we easily see from the proof below), we have the following result.

Theorem 6.5 (Griffiths 1969) *For every integer p between 1 and n, the image of the natural map*

$$H^0(X, K_X(pY)) \to H^n(U, \mathbb{C}) \tag{6.4}$$

which to a section α (viewed as a meromorphic form on X of degree n, and therefore closed, holomorphic on U and having a pole of order p along Y) associates its de Rham cohomology class, is equal to $F^{n-p+1}H^n(U)$.

We begin by proving the following lemma.

Lemma 6.6 *Let α be a meromorphic closed differential form of degree k defined on an open set V of X, holomorphic outside $V \cap Y$ and having a pole of order l along $V \cap Y$. Then if $l \geq 2$, locally in V we can write*

$$\alpha = d\beta + \gamma,$$

where β and γ are meromorphic with a pole of order at most $l-1$ along $V \cap Y$, and holomorphic outside Y.

If $l = 1$, then α is a logarithmic form.

Proof The last assertion is a consequence of the following characterisation of forms with logarithmic singularities along Y: they are those which have a pole of order at most 1 and whose exterior differential has a pole of order at most 1 along Y (see vI.8.2.2).

Assume then that $l \geq 2$, and let us write

$$\alpha = \frac{dz_1 \wedge \alpha'}{z_1^l} + \frac{\alpha''}{z_1^l},$$

locally, where we have chosen holomorphic coordinates $z_1, \ldots, z_n$ such that Y is defined by $z_1 = 0$, and the forms α' and α'' are holomorphic and do not contain dz_1.

Considering the order of the pole, the fact that $d\alpha = 0$ then immediately implies that the form α'' vanishes along $z_1 = 0$, so that $\frac{\alpha''}{z_1^l}$ has a pole of order at most $l-1$ along Y.

Moreover, let $\beta = -\frac{\alpha'}{(l-1)z_1^{l-1}}$. Then clearly

$$\frac{dz_1 \wedge \alpha'}{z_1^l} = d\beta$$

modulo a form having a pole of order at most $l-1$ along Y. □

Corollary 6.7 *If moreover the degree k of α is at least 2, we can write $\alpha = d\beta$ locally, where β is a meromorphic form of degree $k-1$, having a pole of order at most $l-1$ along Y.*

Proof Reasoning by induction on the order l of the pole, we deduce from the lemma that α can be written $d\beta + \alpha'$, where β is a meromorphic form of degree $k-1$, with a pole of order at most $l-1$ along Y, and α' has logarithmic singularities along Y. But we showed in vI.8.2.3 that the holomorphic logarithmic de Rham complex is locally exact in degree ≥ 2 if the hypersurface Y is smooth. Thus, if $k \geq 2$, then $\alpha' = d\gamma$, where γ has logarithmic singularities along Y. □

Proof of theorem 6.5 Let $\Omega_X^{k,c}(lY)$ denote the sheaf of closed meromorphic differential forms of degree k, holomorphic on U and having a pole of order at most l along Y. Then corollary 6.7 yields the following exact sequences for $l \geq 2,\ k \geq 2$:

$$0 \to \Omega_X^{k-1,c}((l-1)Y) \to \Omega_X^{k-1}((l-1)Y) \xrightarrow{d} \Omega_X^{k,c}(lY) \to 0. \tag{6.5}$$

Finally, if $l = 1$, we have the equality

$$\Omega_X^{k,c}(\log Y) = \Omega_X^{k,c}(lY). \tag{6.6}$$

Starting from $K_X(pY) = \Omega_X^{n,c}(pY)$, and (if $p \geq 2$) applying (6.5) repeatedly, then because $n - p + 2 \geq 2$, we obtain

$$0 \to \Omega_X^{n-1,c}((p-1)Y) \to \Omega_X^{n-1}((p-1)Y) \xrightarrow{d} K_X(pY) \to 0,$$
$$0 \to \Omega_X^{n-2,c}((p-2)Y) \to \Omega_X^{n-2}((p-2)Y) \xrightarrow{d} \Omega_X^{n-1,c}((p-1)Y) \to 0,$$
$$\cdots$$
$$0 \to \Omega_X^{n-p+1,c}(\log Y) \to \Omega_X^{n-p+1}(Y) \xrightarrow{d} \Omega_X^{n-p+2,c}(2Y) \to 0.$$

Now consider the long exact sequences of cohomology associated to these short exact sequences. Applying the vanishing hypotheses (*), we obtain surjective maps

$$H^0(X, K_X(pY)) \twoheadrightarrow H^1\big(X, \Omega_X^{n-1,c}((p-1)Y)\big),$$
$$H^1\big(X, \Omega_X^{n-1,c}((p-1)Y)\big) \twoheadrightarrow H^2\big(X, \Omega_X^{n-2,c}((p-2)Y)\big),$$
$$\cdots$$
$$H^{p-2}\big(X, \Omega_X^{n-p+2,c}(2Y)\big) \twoheadrightarrow H^{p-1}\big(X, \Omega_X^{n-p+1,c}(\log Y)\big).$$

To conclude the proof of the theorem, note that by the definition of the Hodge filtration on $H^n(U, \mathbb{C})$, we have

$$H^{p-1}\big(X, \Omega_X^{n-p+1,c}(\log Y)\big) = F^{n-p+1}H^n(U, \mathbb{C}).$$

Indeed, the logarithmic de Rham complex is exact in degree ≥ 2, so for $k \geq 1$, the complex

$$F^k\Omega_X^\bullet(\log Y) = 0 \to \Omega_X^k(\log Y) \to \Omega_X^{k+1}(\log Y) \to \cdots$$

is a resolution of the sheaf $\Omega_X^{k,c}(\log Y)$. As $n - p + 1 \geq 1$, we obtain

$$H^{p-1}\big(X, \Omega_X^{n-p+1,c}(\log Y)\big) = \mathbb{H}^n(F^{n-p+1}\Omega_X^\bullet(\log Y)).$$

Now, by definition, we have

$$F^{n-p+1}H^n(U, \mathbb{C}) = \operatorname{Im}(\mathbb{H}^n(F^{n-p+1}\Omega_X^\bullet(\log Y)) \to \mathbb{H}^n(\Omega_X^\bullet(\log Y))),$$

and this last map is injective by the degeneracy at E_1 of the Frölicher spectral sequence (see vI.8.4.3).

https://doi.org/10.1017/CBO9780511615177.007 Published online by Cambridge University Press

We have thus obtained a surjective map

$$H^0(X, K_X(pY)) \twoheadrightarrow F^{n-p+1}H^n(U, \mathbb{C}),$$

and so all we have to do is check that this map is indeed the map which to a meromorphic form of degree n, holomorphic on U, associates its de Rham cohomology class. But this is easy. $\square$

6.1.3 The case of hypersurfaces of $\mathbb{P}^n$

Now assume that X is the projective space $\mathbb{P}^n$, and that Y is a smooth hypersurface of degree d, with equation $f = 0$. We know that $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, a generator of $H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(n+1))$ being given by

$$\begin{aligned}\Omega &= \sum_i (-1)^i X_i dX_0 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_n \\ &= X_0 \ldots X_n \sum_i (-1)^i \frac{dX_0}{X_0} \wedge \cdots \wedge \frac{d\hat{X}_i}{X_i} \wedge \cdots \wedge \frac{dX_n}{X_n},\end{aligned}$$

where the X_i are homogeneous coordinates on $\mathbb{P}^n$. As $\mathcal{O}_{\mathbb{P}^n}(Y) = \mathcal{O}_{\mathbb{P}^n}(d)$, theorem 6.5 shows that for every p, $1 \leq p \leq n$, we have a surjective map

$$\alpha_p : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(pd - n - 1)) \to F^{n-p+1}H^n(U, \mathbb{C}) \cong F^{n-p}H^{n-1}(Y, \mathbb{C})_{\text{van}},$$

which to a polynomial P associates the residue of the class of the meromorphic form $\frac{P\Omega}{f^p}$. Here, the last isomorphism follows from the exact sequence (6.3) and the fact that $H^n(\mathbb{P}^n, \mathbb{C})_{\text{prim}} = 0$.

Remark 6.8 *For hypersurfaces in projective space, the vanishing middle degree cohomology is the same as the primitive cohomology. Hence we shall use the notation $H^*(Y)_{\text{prim}}$, which is more common.*

Definition 6.9 *The Jacobian ideal $J_f = \bigoplus J_f^l$ of f is the homogeneous ideal of the ring of polynomials*

$$S = \bigoplus\nolimits_l S^l, \quad S^l = \mathrm{Sym}^l S^1 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l))$$

generated by the partial derivatives $\frac{\partial f}{\partial X_i}$, $i = 0, \ldots, n$.

Clearly, this ideal does not depend on the choice of coordinates, since every other coordinate system is obtained by applying a linear transformation to the coordinates X_i.

Theorem 6.10 (Griffiths 1969) *The kernel of the composed map*

$$\overline{\alpha}_p : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(pd - n - 1)) \to F^{n-p}H^{n-1}(Y, \mathbb{C})$$
$$\to F^{n-p}H^{n-1}(Y, \mathbb{C})/F^{n-p+1}H^{n-1}(Y, \mathbb{C}) = H^{n-p,p-1}(Y) \tag{6.7}$$

is equal to J_f^{pd-n-1}.

Proof We know that the image of the map α_{p-1} is $F^{n-p+1}H^{n-1}(Y, \mathbb{C})_{\mathrm{prim}}$, so we have

$$P \in \operatorname{Ker} \overline{\alpha}_p \Leftrightarrow \exists Q \in S^{(p-1)d-n-1}, \quad \alpha_{p-1}(Q) = \alpha_p(P).$$

As we clearly have

$$\alpha_{p-1}(Q) = \alpha_p(f Q),$$

this is equivalent to $P - f Q \in \operatorname{Ker} \alpha_p$.

Now, the kernel of α_p is in fact described in the proof of theorem 6.5. Indeed, as the vanishing hypotheses (*) are satisfied, we find that the maps

$$H^1\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1,c}((p-1)Y)\big) \twoheadrightarrow H^2\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-2,c}((p-2)Y)\big),$$
$$\cdots$$
$$H^{p-2}\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-p+2,c}(2Y)\big) \twoheadrightarrow H^{p-1}\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-p+1,c}(\log Y)\big)$$

are all isomorphisms. Moreover, as $H^n(\mathbb{P}^n, \mathbb{C})_{\mathrm{prim}} = 0$, the arrow

$$\operatorname{Res} : H^{p-1}\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-p+1,c}(Y)\big) = F^{n-p+1}H^n(U, \mathbb{C}) \to F^{n-p}H^{n-1}(Y, \mathbb{C})_{\mathrm{prim}}$$

is also an isomorphism. It follows that $\operatorname{Ker} \alpha_p$ is equal to the kernel of the map

$$H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(pY)) \twoheadrightarrow H^1\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1,c}((p-1)Y)\big),$$

or also, by the exact sequence

$$0 \to \Omega_{\mathbb{P}^n}^{n-1,c}((p-1)Y) \to \Omega_{\mathbb{P}^n}^{n-1}((p-1)Y) \xrightarrow{d} K_{\mathbb{P}^n}(pY) \to 0,$$

to $dH^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}((p-1)Y))$. The theorem is thus equivalent to the following statement.

Lemma 6.11 *Let* $P \in S^{pd-n-1}$. *Then there exists* $Q \in S^{(p-1)d-n-1}$ *such that*

$$\frac{(P - f Q)}{f^p}\Omega = d\gamma, \quad \gamma \in H^0\big(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}((p-1)d)\big)$$

if and only if $P \in J_f^{pd-n-1}$.

Proof of lemma 6.11 Let $\gamma \in H^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)d))$. γ is considered as a meromorphic form of degree $n-1$ on $\mathbb{P}^n$, holomorphic on U and having a pole of order at most $p-1$ along Y. Moreover, by the interior product, we have a canonical isomorphism

$$\mathrm{int}()(\Omega) : T_{\mathbb{P}^n} \to \Omega^{n-1}_{\mathbb{P}^n}(n+1),$$

and thus an isomorphism

$$\mathrm{int}()(\Omega) : H^0(\mathbb{P}^n, T_{\mathbb{P}^n}((p-1)d-n-1)) \to H^0\big(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((p-1)d)\big).$$

Finally, we have the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to V \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to T_{\mathbb{P}^n} \to 0, \tag{6.8}$$

where $V = (S^1)^*$ is generated by the $\frac{\partial}{\partial X_i}$, $i = 0, \dots, n$. Here, we consider $\frac{\partial}{\partial X_i}$ as a vector field on $\mathbb{C}^{n+1}$, which does not descend to a vector field on $\mathbb{P}^n$, whereas the vector fields $X_j \frac{\partial}{\partial X_i}$ on $\mathbb{C}^{n+1}$ do descend to vector fields on $\mathbb{P}^n$. The Euler vector field $E = \sum_i X_i \frac{\partial}{\partial X_i}$ generates the kernel of the exact sequence (6.8), as it is the vector field tangent to the fibres of

$$\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n.$$

From the Euler exact sequence together with the fact that $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for $k \geq 0$, we deduce that $H^0(\mathbb{P}^n, T_{\mathbb{P}^n}((p-1)d-n-1))$ is generated by the vector fields $\sum_i P_i \frac{\partial}{\partial X_i}$ with

$$P_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((p-1)d-n)) = S^{(p-1)d-n}.$$

Thus, γ can be written

$$\frac{\sum_i P_i \,\mathrm{int}\big(\frac{\partial}{\partial X_i}\big)(\Omega)}{f^{p-1}},$$

with P_i arbitrary in $S^{(p-1)d-n}$. (More precisely, this expression should be understood as that of the pullback of γ on $\mathbb{C}^{n+1} - \{0\}$.)

We thus obtain

$$d\gamma = -(p-1)\frac{\sum_{i,j} P_i \frac{\partial f}{\partial X_j} dX_j \wedge \mathrm{int}\big(\frac{\partial}{\partial X_i}\big)(\Omega)}{f^p} + \frac{d\Big(\sum_i P_i \,\mathrm{int}\big(\frac{\partial}{\partial X_i}\big)(\Omega)\Big)}{f^{p-1}}.$$

Clearly, the second term in this expression is a meromorphic form with a pole of order $\leq p-1$ along Y. Recalling also that

$$\Omega = \sum_i (-1)^i X_i dX_0 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_n,$$

we find

$$dX_j \wedge \operatorname{int}\left(\frac{\partial}{\partial X_i}\right)(\Omega) = -\operatorname{int}\left(\frac{\partial}{\partial X_i}\right)(dX_j \wedge \Omega) + \delta_{ij}\Omega$$
$$= -\operatorname{int}\left(\frac{\partial}{\partial X_i}\right)(X_j dX_0 \wedge \cdots \wedge dX_n) + \delta_{ij}\Omega.$$

Finally, we obtain

$$d\gamma = -\frac{(p-1)}{f^p}\left(\sum_i P_i \frac{\partial f}{\partial X_i}\right)\Omega$$
$$+\frac{(p-1)}{f^p}\sum_{i,j} P_i X_j \frac{\partial f}{\partial X_j}\operatorname{int}\left(\frac{\partial}{\partial X_i}\right)(dX_0 \wedge \cdots \wedge dX_n)$$

modulo a meromorphic form having a pole of order at most $p-1$ along Y. Then, using the Euler relation

$$\sum_j X_j \frac{\partial f}{\partial X_j} = \deg(f) f,$$

we see that the last term admits a pole of order at most $p-1$ along Y, so that

$$d\gamma = -\frac{(p-1)}{f^p}\sum_i P_i \frac{\partial f}{\partial X_i}\Omega$$

modulo a meromorphic form having a pole of order at most $p-1$ along Y.

As the P_i are arbitrary, we deduce that a form $\frac{P\Omega}{f^p}$ can be written $d\gamma$ modulo a form having a pole of order at most $p-1$ along Y if and only if $P \in J_f^{pd-n-1}$. This proves the lemma, and thus concludes the proof of theorem 6.10. □

As the map $\overline{\alpha}_p : S^{pd-n-1} \to F^{n-p}H^{n-1}(Y,\mathbb{C})_{\text{prim}}/F^{n-p+1}H^{n-1}(Y,\mathbb{C})_{\text{prim}}$ is surjective by theorem 6.5, and moreover

$$F^{n-p}H^{n-1}(Y,\mathbb{C})_{\text{prim}}/F^{n-p+1}H^{n-1}(Y,\mathbb{C})_{\text{prim}} = H^{n-p,p-1}(Y)_{\text{prim}}$$
$$:= H^{n-p,p-1}(Y) \cap H^{n-1}(Y,\mathbb{C})_{\text{prim}},$$

we deduce from theorem 6.10 the following description of the primitive Dolbeault cohomology groups of Y.

Corollary 6.12 *The residue map induces a natural isomorphism*

$$R_f^{pd-n-1} \cong H^{n-p,p-1}(Y)_{\text{prim}},$$

where $R_f^l := S^l/J_f^l$ denotes the lth component of the Jacobian ring $R_f = S/J_f$.

6.2 IVHS of hypersurfaces

6.2.1 Computation of $\overline{\nabla}$

Let X be an n-dimensional projective variety, and let $Y \subset X$ be a hypersurface satisfying the conditions (*) of 6.1.2. Let $B \subset H^0(X, \mathcal{O}_X(Y))$ denote the Zariski open set consisting of the polynomials f such that the hypersurface Y with equation $f = 0$ is smooth. We have the universal smooth hypersurface

$$\pi : \mathcal{Y} \to B,$$

and we will describe the infinitesimal variation of Hodge structure on $H^{n-1}(Y, \mathbb{C})_{\mathrm{van}}$ for $f \in B$, i.e. the maps

$$\overline{\nabla}_{l,f} : H^{l,n-1-l}(Y)_{\mathrm{van}} \to \mathrm{Hom}\,(T_{B,f}, H^{l-1,n-l}(Y)_{\mathrm{van}}).$$

By theorem 6.5, we have the surjective maps of (6.7)

$$\overline{\alpha}_p : H^0(X, K_X(pY)) \to H^{n-p,p-1}(Y)_{\mathrm{van}}.$$

We have the following result due to Carlson & Griffiths (1980).

Theorem 6.13 *The map*

$$\overline{\nabla}_{n-p,f} : H^{n-p,p-1}(Y)_{\mathrm{van}} \to \mathrm{Hom}\,(T_{B,f}, H^{n-p-1,p}(Y)_{\mathrm{van}})$$

can be described as follows. For $P \in H^0(X, K_X(pY))$ and $H \in T_{B,f} = H^0(X, \mathcal{O}_X(Y))$, we have

$$\overline{\nabla}_{n-p,f}(\overline{\alpha}_p(P))(H) = -p\overline{\alpha}_{p+1}(PH).$$

In other words, the diagram

$$\begin{array}{lccc} & H^0(X, K_X(pY)) & \longrightarrow & \mathrm{Hom}\,(H^0(X, \mathcal{O}_X(Y)), H^0(X, K_X((p+1)Y))) \\ & \downarrow & & \downarrow \\ \overline{\nabla}: & H^{n-p,p-1}(Y)_{\mathrm{van}} & \longrightarrow & \mathrm{Hom}\,(T_{B,f}, H^{n-p-1,p}(Y)_{\mathrm{van}}), \end{array}$$

where the upper horizontal arrow is given by multiplication, and the vertical arrows are given by $\overline{\alpha}_p$, $\overline{\alpha}_{p+1}$ and the identification $T_{B,f} = H^0(X, \mathcal{O}_X(Y))$, is commutative up to a multiplicative coefficient.

Proof Note that B also parametrises the universal open set

$$\pi_U : \mathcal{U} \to B, \quad \mathcal{U} := X \times B - \mathcal{Y}.$$

We showed earlier that the residue induces a morphism of local systems

$$\mathrm{Res} : (R^n \pi_U)_* \mathbb{Z} \twoheadrightarrow R^{n-1}\pi_* \mathbb{Z}_{\mathrm{van}}.$$

Writing

$$\mathcal{H}_U^n, \ \ \text{resp.}\ \mathcal{H}_{\mathrm{van}}^{n-1}$$

for the corresponding flat vector bundles, the residue map then commutes with the Gauss–Manin connections, which we denote by ∇_U and ∇ respectively.

Recall that if $\lambda \in H^{n-p,p-1}(Y)_{\mathrm{van}}$ and $H \in T_{B,f}$, then

$$\overline{\nabla}_H(\lambda) \in H^{n-p-1,p}(Y)_{\mathrm{van}}$$

is obtained by choosing a section $\tilde{\lambda} \in F^{n-p}\mathcal{H}_{\mathrm{van}}^{n-1}$ such that the image of $\tilde{\lambda}(f)$ in

$$\left(F^{n-p}\mathcal{H}_{\mathrm{van}}^{n-1}/F^{n-p+1}\mathcal{H}_{\mathrm{van}}^{n-1}\right)_f = H^{n-p,p-1}(Y)_{\mathrm{van}}$$

is equal to λ. We then have

$$\overline{\nabla}_f(\lambda)(H) = \nabla_H \tilde{\lambda} \bmod \left(F^{n-p}\mathcal{H}_{\mathrm{van}}^{n-1}\right)_f.$$

Now let $P \in H^0(X, K_X(pY))$ be such that $\overline{\alpha}_p(P) = \lambda$. Here, we identify $K_X(pY)$ with $K_X \otimes \mathcal{L}^{\otimes p}$, $\mathcal{L} = \mathcal{O}_X(Y)$. Then $f \in H^0(X, \mathcal{L})$ and $\frac{P\Omega}{f^p}$ is a meromorphic form with a pole of order p along Y. By theorem 6.5, the section

$$g \mapsto \mathrm{Res}_{Y_g}\left[\frac{P\Omega}{g^p}\right], \quad g \in B$$

of $\mathcal{H}_{\mathrm{van}}^{n-1}$, where the bracket denotes the de Rham cohomology class of the closed form considered, is a section of $F^{n-p}\mathcal{H}_{\mathrm{van}}^{n-1}$ whose value at the point f projects to λ. It follows that

$$\begin{aligned}\overline{\nabla}_f(\lambda)(H) &= \nabla_H\left(\mathrm{Res}_{Y_g}\left[\frac{P\Omega}{g^p}\right]\right) \bmod F^{n-p}H^{n-1}(Y)_{\mathrm{van}}\\ &= \mathrm{Res}_Y \nabla_{U,H}\left(\left[\frac{P\Omega}{g^p}\right]\right) \bmod F^{n-p}H^{n-1}(Y)_{\mathrm{van}},\end{aligned}$$

where $g \mapsto [\frac{P\Omega}{g^p}]$ is a section of $\mathcal{H}_U^n$.

Lemma 6.14 *Let $\omega = (\omega_g)_{g\in B}$ be a family of closed singular differential forms on X, where ω_g is $\mathcal{C}^\infty$ on U_g and varies holomorphically with g. Then* *the corresponding section $\sigma \in \mathcal{H}_U^n$ defined by $\sigma_g = [\omega_g]$ is holomorphic, and we have*

$$\nabla_{U,H}\sigma = [d_H\omega],$$

where d_H is the derivative with respect to the tangent vector $H \in T_{B,f}$.

Temporarily admitting this lemma, we conclude that

$$\begin{aligned}\overline{\nabla}_f(\lambda)(H) &= \mathrm{Res}_Y \nabla_{U,H}\left(\left[\frac{P}{g^p}\right]\right) \bmod F^{n-p}H^{n-1}(Y)_{\mathrm{van}}\\ &= \mathrm{Res}_Y\left[d_H\frac{P}{g^p}\right] = \mathrm{Res}_Y\left(\left[\frac{-pPH}{f^{p+1}}\right]\right) \bmod F^{n-p}H^{n-1}(Y)_{\mathrm{van}},\end{aligned}$$

which proves theorem 6.13. □

Proof of lemma 6.14 The statement is local, so we can replace B by a small open set W containing f. We can then find a tubular neighbourhood T of Y which retracts by deformation to Y_g, for every $g \in W$. Then $X - T$ has the homotopy type of $U_g = X - Y_g$ for every g, and in the statement, we can replace the family of open sets

$$\pi_U : \pi_U^{-1}(W) \to W$$

by the constant open set $(X - T) \times W$. But then the lemma simply states that for a family of closed forms $(\omega_g)_{g\in W}$ on the constant variety $X - T$, and for a tangent vector $H \in T_{W,f}$, we have

$$d_H([\omega_g]) = [d_H\omega_g],$$

which is obvious. □

Now consider the case where $X = \mathbb{P}^n$ and B is the family of polynomials of degree d parametrising smooth hypersurfaces.

Lemma 6.15 *If $n \geq 2$, the kernel of the Kodaira–Spencer map*

$$\rho : T_{B,f} = S^d \to H^1(Y, T_Y)$$

is J_f^d. Moreover, this map is surjective if $n \geq 4$, or $n = 3,\ d \neq 4$.

Proof By vI.9.1.2, we know that ρ is the connecting morphism associated to the exact sequence

$$0 \to T_Y \to T_{\mathcal{Y}|Y} \to \pi^* T_{B,f} \to 0,$$

where $Y \subset \mathcal{Y}$ is identified with the fibre $\pi^{-1}(f)$. Now, as $\mathcal{Y} \subset \mathbb{P}^n \times B$, the map

$$\mathrm{pr}_{1*} : T_{\mathcal{Y}|Y} \to T_{\mathbb{P}^n|Y}$$

gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_Y & \longrightarrow & T_{\mathcal{Y}|Y} & \longrightarrow & \pi^* T_{B,f} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_Y & \longrightarrow & T_{\mathbb{P}^n|Y} & \longrightarrow & \mathcal{O}_Y(d) & \longrightarrow & 0, \end{array} \tag{6.9}$$

in which the lower exact sequence is the normal exact sequence of Y in $\mathbb{P}^n$. Here, the last vertical arrow is simply the evaluation arrow

$$S^d \otimes \mathcal{O}_Y \to \mathcal{O}_Y(d).$$

The commutativity of this diagram shows that ρ is also the composed map

$$S^d \to H^0(\mathcal{O}_Y(d)) \xrightarrow{\overline{\rho}} H^1(Y, T_Y),$$

where the first arrow is surjective and $\overline{\rho}$ is the connecting arrow induced by the normal exact sequence. It follows immediately that

$$\mathrm{Ker}\,\overline{\rho} = \mathrm{Im}\, H^0(Y, T_{\mathbb{P}^n|Y}),$$

and as the restriction map

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \to H^0(Y, T_{\mathbb{P}^n|Y})$$

is surjective for $n \geq 2$ by Bott's theorem (Bott 1957), and the composed map

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \to H^0(Y, T_{\mathbb{P}^n|Y}) \to H^0(Y, \mathcal{O}_Y(d))$$

sends $X_i \frac{\partial}{\partial X_j}$ to $X_i \frac{\partial f}{\partial X_j}{}_{|Y}$ by definition of the normal exact sequence, we conclude that $\mathrm{Ker}\,\rho$ is generated by f and J^d_f. Finally, Euler's relation shows that $f \in J^d_f$, so $\mathrm{Ker}\,\rho$ is generated by J^d_f. The assertion concerning the surjectivity follows from the above argument, since by Bott's vanishing theorem, we have $H^1(Y, T_{\mathbb{P}^n|Y}) = 0$ for $n \geq 4$ or $n = 3,\ d \neq 4$. □

Remark 6.16 *Note that J^d_f is the tangent space at the point f of the orbit O_f of f under the action of the group* Gl$(n+1)$. *Indeed, the tangent space to O_f is generated by the $df_t/dt_{|t=0}$, where*

$$f_t = g_t^*(f), \quad g_t = \mathrm{Id} + tA, \quad A \in M^{n+1}(\mathbb{C}).$$

Writing $g_t^* X_i = X_i + t\sum_j A_{i,j} X_j$, *we find*

$$g_t^* f = f(X_0 + tA_0, \ldots, X_n + tA_n), \quad A_i = \sum_j A_{i,j} X_j.$$

Thus, we have

$$\frac{d}{dt}(f_t)_{|t=0} = \sum_i A_i \frac{\partial f}{\partial X_i},$$

which shows that $T_{O_f,f} = J_f^d$.

If B' *is the quotient of* B^0 *by* $\mathrm{Gl}(n+1)$, *where* B^0 *is the open set of* B *parametrising the hypersurfaces without any non-trivial automorphisms, and* $\pi : \mathcal{Y}' \to B'$ *is the universal quotient hypersurface of* $\mathcal{Y}$ *by the action of* $\mathrm{Gl}(n+1)$ *on* $\mathbb{P}^n \times B$, *then lemma 6.15 shows that* $\mathcal{Y}'$ *is a universal deformation of each of the fibres* Y_f *of* π *for* $n \geq 4$ *or* $n = 3,\ d \neq 4$.

Theorem 6.13 can now be reformulated for hypersurfaces of $\mathbb{P}^n$ in the following way. The infinitesimal variation of Hodge structure for the family $\pi : \mathcal{Y}' \to B'$ is given by the maps

$$\overline{\nabla}_{n-p} : H^{n-p,p-1}(Y)_{\mathrm{prim}} \to \mathrm{Hom}\,(T_{B',f}, H^{n-p-1,p}(Y)_{\mathrm{prim}}),$$

and we have the following result.

Theorem 6.17 *Via the isomorphisms*

$$\overline{\alpha}_p : R_f^{pd-n-1} \cong H^{n-p,p-1}(Y)_{\mathrm{prim}}, \quad \overline{\alpha}_{p+1} : R_f^{(p+1)d-n-1} \cong H^{n-p-1,p}(Y)_{\mathrm{prim}}$$

of corollary 6.12, together with the isomorphism $R_f^d \cong T_{B',f}$, $\overline{\nabla}_{n-p}$ *can be identified up to a multiplicative coefficient with the map given by the product*

$$R_f^{pd-n-1} \to \mathrm{Hom}\left(R_f^d, R_f^{(p+1)d-n-1}\right).$$

6.2.2 Macaulay's theorem

The Jacobian ideal of a hypersurface $Y \subset \mathbb{P}^n$ is generated in the ring of polynomials S by the partial derivatives $G_i = \frac{\partial f}{\partial X_i},\ i = 0, \ldots, n$ of the equation f of Y. The G_i have no common zeros in $\mathbb{P}^n$ when Y is smooth. Indeed, by Euler's relation, if $\frac{\partial f}{\partial X_i}(x) = 0$ for all i, we also have $f(x) = 0$, so $x \in Y$ is a singular point of Y.

Definition 6.18 *A sequence of homogeneous polynomials $G_i \in S^{d_i}$, $i = 0, \dots, n$, with $d_i > 0$, is said to be regular if the G_i have no common zero.*

Given such a regular sequence, let $R_{G.}$ denote the quotient ring $S/J_{G.}$, where $J_{G.}$ is the ideal generated by the G_i. We know by Hilbert's Nullstellensatz that $J^k_{G.} = S^k$ for sufficiently large k, so that $R_{G.}$ is an Artinian ring.

Theorem 6.19 (Macaulay) *The ring $R_{G.}$ satisfies the following property: for $N = \sum_{i=0}^n d_i - n - 1$, we have $\operatorname{rank} R^N_{G.} = 1$, and for every integer k, the pairing*

$$R^k_{G.} \times R^{N-k}_{G.} \to R^N_{G.} \tag{6.10}$$

is perfect.

Such a ring is called a graded Gorenstein ring. The 1-dimensional vector space $R^N_{G.}$ is called its socle. As a consequence of the duality (6.10), we have the following fact.

Corollary 6.20 *We have*

(i) $R^k_{G.} \neq 0 \Leftrightarrow 0 \leq k \leq N$.
(ii) For integers a, b such that $b \geq 0$ and $a + b \leq N$, the map given by the product

$$\mu : R^a_{G.} \to \operatorname{Hom}\left(R^b_{G.}, R^{b+a}_{G.}\right)$$

is injective.

Proof If $k > N$ we have $N - k < 0$, so $R^{N-k}_{G.} = 0$, i.e. $R^k_{G.} = 0$ by duality. Conversely, if $R^k_{G.} = 0$ for $k \geq 0$, then $R^l_{G.} = 0$ for all $l \geq k$, so $N < k$, which proves (i).

Let us now prove (ii). Let $A \in \operatorname{Ker} \mu \subset R^a_{G.}$. For every $B \in R^b_{G.}$, we have $AB = 0$ in $R^{b+a}_{G.}$. Thus, for every $\phi \in R^{N-a-b}_{G.}$, we have $AB\phi = 0 \in R^N_{G.}$. Now, as $b \geq 0$ and $N - a - b \geq 0$, the product

$$\begin{aligned} R^{N-a-b}_{G.} \otimes R^b_{G.} &\to R^{N-a}_{G.}, \\ \phi \otimes B &\mapsto \phi B \end{aligned}$$

is surjective. Thus, $AC = 0$ in $R^N_{G.}$ for every $C \in R^{N-a}_{G.}$, and by Macaulay's theorem, this implies that $A = 0$. □

Proof of theorem 6.19 Let $\mathcal{E}$ denote the holomorphic vector bundle $\bigoplus_i \mathcal{O}_{\mathbb{P}^n}(-d_i)$ on $\mathbb{P}^n$. Every G_i can be viewed as a morphism $G_i : \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{O}_{\mathbb{P}^n}$, and we will write $G_\cdot : \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}$ for the sum of these morphisms. The fact that the G_i have no common zero is equivalent to the surjectivity of the morphism $G_\cdot$. Moreover, by definition, the ideal $J_{G_\cdot}$ satisfies

$$J^k_{G_\cdot} = \operatorname{Im} G_\cdot : H^0(\mathbb{P}^n, \mathcal{E}(k)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)).$$

As the morphism $G_\cdot$ is a section of the dual of the bundle $\mathcal{E}$, by contraction it also induces a morphism

$$G^l_\cdot : \bigwedge^l \mathcal{E} \to \bigwedge^{l-1} \mathcal{E}$$

for each l. For $l = 1$, we have $G^l_\cdot = G_\cdot$; furthermore it is clear that $G^{l-1}_\cdot \circ G^l_\cdot = 0$, and the complex $\mathcal{E}^\cdot$ given by

$$0 \to \bigwedge^{n+1} \mathcal{E} \xrightarrow{G^{n+1}_\cdot} \bigwedge^n \mathcal{E} \to \cdots \to \mathcal{E} \xrightarrow{G_\cdot} \mathcal{O}_{\mathbb{P}^n} \to 0 \tag{6.11}$$

is exact. Indeed, these assertions can be proved pointwise. At a given point $x \in \mathbb{P}^n$, the fibre $\mathcal{E}_x$ is a vector space V, and the morphism $G_\cdot$ is a linear form $\sigma \in V^*$. We then use the fact that the interior product $\sigma_l : \bigwedge^l V \to \bigwedge^{l-1} V$ by σ satisfies $\sigma_l \circ \sigma_{l-1} = 0$, and the sequence

$$\bigwedge^{l+1} V \xrightarrow{\sigma_{l+1}} \bigwedge^l V \xrightarrow{\sigma_l} \bigwedge^{l-1} V$$

is exact at the middle for every l, which is an easy result of linear algebra. The complex (6.11) is called the Koszul resolution associated to $(\mathcal{E}, G_\cdot)$.

Let us assign the degree 0 to the first term $\bigwedge^{n+1} \mathcal{E}$. The twisted complex $\mathcal{E}^\cdot(k)$ is exact, so in particular we have $\mathbb{H}^{n+1}(\mathbb{P}^n, \mathcal{E}^\cdot(k)) = 0$ for all k. But in the hypercohomology spectral sequence of this complex associated to the naive filtration

$$F^p \mathcal{E}^\cdot(k) = 0 \to \mathcal{E}^p(k) \to \cdots \to \mathcal{E}^{n+1}(k) \to 0,$$

the term $E_1^{p,q}$ is given by $E_1^{p,q} = H^q(\mathbb{P}^n, \mathcal{E}^p(k))$, so by Bott's vanishing theorem, since the $\mathcal{E}^p$ are direct sums of line bundles, we have $E_1^{p,q} = 0$ for $q \neq 0, n$. Thus we are actually considering a spherical spectral sequence (see subsection 4.1.3), and we have $E_2^{p,q} = \cdots = E_{n+1}^{p,q}$ and

$$\begin{aligned} E_\infty^{p,0} &= \operatorname{Coker}\left(d_{n+1} : E_2^{p-n-1,n} \to E_2^{p,0}\right), \\ E_\infty^{p-n-1,n} &= \operatorname{Ker}\left(d_{n+1} : E_2^{p-n-1,n} \to E_2^{p,0}\right), \\ E_\infty^{p,q} &= 0,\ q \neq 0, n. \end{aligned}$$

As we also know that $\mathbb{H}^l(\mathbb{P}^n, \mathcal{E}^{\cdot}(k)) = 0$ for all l, we have in particular $E_\infty^{p-n-1,n} = E_\infty^{p,0} = 0$, so that the arrows d_{n+1} above are isomorphisms.

But since d_1 is induced by the differential of $\mathcal{E}^{\cdot}(k)$, we also have

$$\begin{aligned} E_2^{n+1,0} &= H^0(\mathbb{P}^n, \mathcal{E}^{n+1}(k))/G.H^0(\mathbb{P}^n, \mathcal{E}^n(k)) \\ &= H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))/G.H^0(\mathbb{P}^n, \mathcal{E}(k)) = R^k_{G.}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E_2^{0,n} &= \operatorname{Ker}(G. : H^n(\mathbb{P}^n, \mathcal{E}^0(k)) \to H^n(\mathbb{P}^n, \mathcal{E}^1(k))) \\ &= \operatorname{Ker}\left(G. : H^n\left(\mathbb{P}^n, \bigwedge^{n+1} \mathcal{E}(k)\right) \to H^n\left(\mathbb{P}^n, \bigwedge^{n} \mathcal{E}(k)\right)\right). \end{aligned}$$

We have thus constructed a canonical isomorphism

$$d_{n+1} : \operatorname{Ker}\left(G. : H^n\left(\mathbb{P}^n, \bigwedge^{n+1} \mathcal{E}(k)\right) \to H^n\left(\mathbb{P}^n, \bigwedge^{n} \mathcal{E}(k)\right)\right) \to R^k_{G.}.$$

Now, $\bigwedge^{n+1} \mathcal{E}(k) = \mathcal{O}_{\mathbb{P}^n}(-\sum_i d_i + k)$ and

$$\bigwedge^{n} \mathcal{E}(k) = \operatorname{Hom}\left(\mathcal{E}, \bigwedge^{n+1} \mathcal{E}(k)\right) = \mathcal{E}^*\left(-\sum_i d_i + k\right).$$

One easily checks that via Serre duality, the map

$$G. : H^n\left(\mathbb{P}^n, \bigwedge^{n+1} \mathcal{E}(k)\right) \to H^n\left(\mathbb{P}^n, \bigwedge^{n} \mathcal{E}(k)\right)$$

is the map dual to

$$\begin{aligned} G. : H^0\left(\mathbb{P}^n, \mathcal{E}\left(\sum_i d_i - k - n - 1\right)\right) &\to \\ H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(\sum_i d_i - k - n - 1\right)\right)&, \end{aligned}$$

so that we have constructed an isomorphism

$$d_{n+1} : \left(R_{G.}^{\sum_i d_i - k - n - 1}\right)^* \cong R^k_{G.}. \tag{6.12}$$

In particular, setting $k = 0$, we find that $R^N_{G.}$ is of rank 1 for $N = \sum_i d_i - n - 1$. Finally, to conclude the proof of theorem 6.19, it suffices to note that the isomorphisms (6.12) satisfy the following compatibility, where the vertical arrows are given by multiplication by $P \in S^l$ and its dual:

$$\begin{array}{lccc} d_{n+1}: & \left(R_{G.}^{N-k}\right)^* & \xrightarrow{\cong} & R_{G.}^k \\ & \Big\downarrow{\scriptstyle P^*} & & \Big\downarrow{\scriptstyle P} \\ d_{n+1}: & \left(R_{G.}^{N-k-l}\right)^* & \xrightarrow{\cong} & R_{G.}^{k+l}. \end{array} \tag{6.13}$$

Indeed, the commutativity of this diagram says that the pairing (6.10) satisfies the property

$$\langle A, PB\rangle = \langle PA, B\rangle$$

for $A \in R_{G.}^{N-k-l}$, $P \in R_{G.}^l$, $B \in R_{G.}^k$, in other words that the isomorphism (6.12) is given by the pairing of (6.10). □

6.2.3 The symmetriser lemma

As above, consider the ring $R_{G.} = S/J_{G.}$ associated to a regular sequence $G_0, \ldots, G_n$ on $\mathbb{P}^n$. Let $d_i = deg\, G_i$ and $N = \sum_i d_i - n - 1$.

As the ring $R_{G.}$ is commutative, if $b \geq a$ are integers and $P \in R_{G.}^{b-a}$, the multiplication by P

$$\mu_P : R_{G.}^a \to R_{G.}^b$$

satisfies the following property:

$$\forall A,\ B \in R_{G.}^a, \quad A\mu_P(B) = B\mu_P(A) \in R_{G.}^{b+a}.$$

Furthermore, Macaulay's theorem 6.19 shows that the map

$$\mu : P \mapsto \mu_P, \quad R_{G.}^{b-a} \to \mathrm{Hom}\left(R_{G.}^a, R_{G.}^b\right)$$

is injective for $b \leq N$, $a \geq 0$. The symmetriser lemma (see Donagi & Green 1984) is the following statement.

Proposition 6.21 *Let*

$$T^{a,b} := \left\{\phi \in \mathrm{Hom}\left(R_{G.}^a, R_{G.}^b\right) \mid A\phi(B) = B\phi(A) \in R_{G.}^{b+a}\ \ \forall A,\ B \in R_{G.}^a\right\}.$$

If $a + b < N$ *and* $\sup_i(d_i + b) \leq N$, *then we have the equality*

$$\mu\left(R_{G.}^{b-a}\right) = T^{a,b} \subset \mathrm{Hom}\left(R_{G.}^a, R_{G.}^b\right).$$

Proof By definition, $T^{a,b}$ is the kernel of the map

$$\sigma : \operatorname{Hom}(R^a_{G.}, R^b_{G.}) \to \operatorname{Hom}\left(\bigwedge^2 R^a_{G.}, R^{a+b}_{G.}\right)$$

given by $\sigma(\phi)(A \wedge B) = A\phi(B) - B\phi(A)$. We want to show that the following sequence is exact at the middle:

$$R^{b-a}_{G.} \xrightarrow{\mu} \operatorname{Hom}\left(R^a_{G.}, R^b_{G.}\right) \xrightarrow{\sigma} \operatorname{Hom}\left(\bigwedge^2 R^a_{G.}, R^{a+b}_{G.}\right). \tag{6.14}$$

Equivalently, consider the dual exact sequence

$$\bigwedge^2 R^a_{G.} \otimes \left(R^{a+b}_{G.}\right)^* \xrightarrow{\sigma^*} R^a_{G.} \otimes \left(R^b_{G.}\right)^* \xrightarrow{\mu^*} \left(R^{b-a}_{G.}\right)^*. \tag{6.15}$$

Applying Macaulay's theorem, we have natural isomorphisms

$$\left(R^b_{G.}\right)^* \cong R^{N-b}_{G.}, \quad \left(R^{a+b}_{G.}\right)^* \cong R^{N-a-b}_{G.}, \quad \left(R^{b-a}_{G.}\right)^* \cong R^{N+a-b}_{G.},$$

so that (6.15) can be written

$$\bigwedge^2 R^a_{G.} \otimes R^{N-a-b}_{G.} \xrightarrow{\sigma^*} R^a_{G.} \otimes R^{N-b}_{G.} \xrightarrow{\mu^*} R^{N-b+a}_{G.}. \tag{6.16}$$

Furthermore, the compatibility of the diagram (6.13), i.e. the fact that the maps given by multiplication by P are self-dual for the Macaulay duality, implies immediately that we have

$$\mu^*(A \otimes B) = A \cdot B, \;\; \sigma^*(A \wedge B \otimes C) = A \otimes (B \cdot C) - B \otimes (A \cdot C).$$

The following result on the Koszul cohomology of projective space is due to Green (1984b). Let S be the ring of polynomials with $n+1$ variables, and for every pair of integers k and l, write

$$\begin{aligned} &\mu^* : S^l \otimes S^k \to S^{k+l}, \;\; \mu^*(A \otimes B) = A \cdot B \\ &\sigma^* : \bigwedge^2 S^l \otimes S^k \to S^l \otimes S^{k+l}, \\ &\qquad \sigma^*(A \wedge B \otimes C) = A \otimes (B \cdot C) - B \otimes (A \cdot C). \end{aligned}$$

Theorem 6.22 (Green) *The sequence*

$$\bigwedge^2 S^l \otimes S^{k-l} \xrightarrow{\sigma^*} S^l \otimes S^k \xrightarrow{\mu^*} S^{k+l}$$

is exact whenever $k > l$.

Consider the case where $l = a$, $k = N - b$. By the hypothesis $N > a + b$, we can apply the theorem to conclude that the sequence

$$\bigwedge^2 S^a \otimes S^{N-a-b} \xrightarrow{\sigma^*} S^a \otimes S^{N-b} \xrightarrow{\mu^*} S^{N-b+a} \tag{6.17}$$

is exact at the middle. Also, we have the following commutative diagram, in which the vertical arrows are surjective:

$$\begin{tikzcd} \bigwedge^2 S^a \otimes S^{N-a-b} \arrow[r, "\sigma^*"] \arrow[d] & S^a \otimes S^{N-b} \arrow[r, "\mu^*"] \arrow[d] & S^{N-b+a} \arrow[d] \\ \bigwedge^2 R^a_{G.} \otimes R^{N-a-b}_{G.} \arrow[r, "\sigma^*"] & R^a_{G.} \otimes R^{N-b}_{G.} \arrow[r, "\mu^*"] & R^{N-b+a}_{G.}. \end{tikzcd} \tag{6.18}$$

Now let $\gamma \in \operatorname{Ker}(\mu^* : R^a_{G.} \otimes R^{N-b}_{G.} \to R^{N-b+a}_{G.})$, and let $\tilde{\gamma}$ be a lifting of γ to $S^a \otimes S^{N-b}$. Then we have $\mu^*(\tilde{\gamma}) \in J^{N-b+a}_{G.}$. But by the hypothesis $N - b \geq \sup(d_i)$, the ideal $J_{G.}$ is generated by its component of degree $N - b$. Thus, we have the surjectivity

$$S^a \otimes J^{N-b}_{G.} \twoheadrightarrow J^{N-b+a}_{G.},$$

which shows that up to modifying the lifting $\tilde{\gamma}$, we may assume that $\mu^*(\tilde{\gamma}) = 0$. The exactness of the sequence (6.17) thus implies that $\tilde{\gamma} \in \operatorname{Im} \sigma^*$, and by the diagram (6.18), we conclude that $\gamma \in \operatorname{Im} \sigma^*$. Hence (6.16) is exact at the middle. □

Remark 6.23 *The same proof also shows that under the same hypotheses on a and b, the sequence*

$$R^{b-a}_{G.} \xrightarrow{\mu} \operatorname{Hom}\left(S^a, R^b_{G.}\right) \xrightarrow{\sigma} \operatorname{Hom}\left(\bigwedge^2 S^a, R^{a+b}_{G.}\right)$$

is exact at the middle.

6.3 First applications

6.3.1 Hodge loci for families of hypersurfaces

Using Macaulay's theorem and the results of section 5.3.2, one can show that the Hodge loci for the family of smooth hypersurfaces of projective space are proper analytic subsets. Precisely, let $\pi : \mathcal{Y} \to B$ be the family of smooth hypersurfaces of degree d of $\mathbb{P}^n$, and let $U \subset B$ be an open set. For every

$$\lambda \in \Gamma(U, R^{n-1}\pi_*\mathbb{C}_{\mathrm{van}})$$

and for every $p \leq n-1$, recall that the corresponding Hodge locus U^p_λ consists of the points $u \in U$ such that $\lambda_u \in F^p H^{n-1}(X_u, \mathbb{C})_{\mathrm{prim}}$. We have the following result (see Carlson *et al.* 1983).

Theorem 6.24 *If p is such that $d(n-p+1)-n-1 \leq (d-2)(n+1)$, then the Hodge loci U_λ^p are proper analytic subsets of U for every $0 \neq \lambda \in \Gamma(U, R^{n-1}\pi_*\mathbb{C}_{\mathrm{prim}})$.*

Macaulay's theorem and its corollary 6.20 say that for $N = (d-2)(n+1)$ and $f \in U$, the Jacobian ring R_f is of rank 1 in degree N, and satisfies $R_f^k \neq 0$ if and only if $0 \leq k \leq N$. By the isomorphism

$$R_f^{(n-p+1)d-n-1} \cong H^{p-1,n-p}(Y_f)_{\mathrm{prim}}$$

of corollary 6.12, the hypothesis is thus equivalent to the condition $H^{p-1,n-p}(Y_f)_{\mathrm{van}} \neq 0$. One shows easily that the condition is equivalent to

$$F^p H^{n-1}(Y_f)_{\mathrm{prim}} \neq H^{n-1}(Y_f)_{\mathrm{prim}}, \quad \forall f \in B,$$

unless $F^p H^{n-1}(Y_f)_{\mathrm{prim}} = 0$. Indeed the description given above of the primitive Dolbeault cohomology of a hypersurface, together with Macaulay's theorem, shows that the set

$$\{p' \in \mathbb{N} \mid H^{p',q'}(Y_f)_{\mathrm{prim}} \neq 0\}$$

is an interval.

The theorem can thus be reformulated as follows.

Theorem 6.25 *If d, n and p are such that $F^p\mathcal{H}^{n-1} \neq \mathcal{H}^{n-1}$, then the Hodge locus U_λ^p is a proper analytic subset of U for every $0 \neq \lambda \in \Gamma(U, R^{n-1}\pi_*\mathbb{C}_{\mathrm{prim}})$.*

Remark 6.26 *Note that if $d \geq n+1$, we have $d(n-p+1)-n-1 \leq (d-2)(n+1)$ for $p \geq 1$, i.e. $F^p\mathcal{H}^{n-1} \neq \mathcal{H}^{n-1}$ for every $p \geq 1$. Theorem 6.25 then says that a class λ which is locally constant on U is generically of maximal Hodge level, i.e. that its component of type $(0, n-1)$ is non-zero.*

Proof of theorem 6.24 Let $f \in U$, and let $\overline{\lambda}_f \in H^{p,n-p-1}(Y_f)_{\mathrm{prim}}$. By theorem 6.13, if $P \in R_f^{(n-p)d-n-1}$ corresponds to $\overline{\lambda}_f$ under the isomorphism

$$\overline{\alpha}_{n-p} : R_f^{(n-p)d-n-1} \cong H^{p,n-p-1}(Y_f)_{\mathrm{prim}},$$

the map $\overline{\nabla}(\overline{\lambda}_f)$ can be identified with the multiplication by P

$$\mu_P : S^d \to R_f^{(n-p+1)d-n-1}.$$

As $(n-p+1)d-n-1 \leq N$, corollary 6.20 shows that $\mu_P = 0$ if and only if $P = 0$. Thus, the map

$$\overline{\nabla}(\overline{\lambda}_f) : T_{U,f} \to H^{p-1,n-p}(Y_f)_{\text{prim}}$$

is non-zero for $0 \neq \overline{\lambda}_f \in H^{p,n-p-1}(Y_f)_{\text{prim}}$. It follows by corollary 5.17 that if $U_\lambda^p = U$, we also have $U_\lambda^{p+1} = U$. Reasoning by induction on $n-p$, we find that if $U_\lambda^p = U$, then $\lambda = 0$. □

Remark 6.27 *Consider the case $n = 3$ and $p = 1$. Then theorem 6.24 says that for $d \geq 4$, the Hodge loci U_λ^1 are proper analytic subsets of U for $\lambda \neq 0$. In particular, this holds for the integral classes λ. Now, we saw that the Hodge loci for the integral classes of degree 2 are the local components of the Noether–Lefschetz locus parametrising surfaces having a holomorphic line bundle which is not a multiple of $\mathcal{O}(1)$. As there is only a countable number of such components, theorem 6.24 gives another, infinitesimal, proof of Noether–Lefschetz theorem 3.32. Theorem 6.24 can thus be viewed as a generalised Noether–Lefschetz theorem.*

Finally, theorem 6.25 can be generalised to the hypersurfaces of high degree of any smooth projective variety X. For this, one uses theorem 6.5 and the analysis of the kernels of the maps $\overline{\alpha}_p$ (see Green 1984c), to obtain the following result.

Theorem 6.28 *Let L be a sufficiently ample invertible bundle on an n-dimensional variety X. Then for every open set $U \subset H^0(X, L)$ and for every locally constant non-zero vanishing class*

$$(\lambda_b)_{b \in U}, \quad \lambda_b \in H^{n-1}(Y_b, \mathbb{C})_{\text{van}},$$

the Hodge loci U_λ^p for $p \geq 1$ are proper analytic subsets of U.

6.3.2 The generic Torelli theorem

Donagi (1983) used the symmetriser lemma to prove a generic Torelli theorem for hypersurfaces of the projective space, with a certain number of exceptions. Some of these, such as cubic surfaces in $\mathbb{P}^3$, are actually counterexamples to the Torelli theorem, whereas others satisfy the statement even though Donagi's method does not apply. This happens in the cases of quartic surfaces in $\mathbb{P}^3$, for which the result is due to Piatetski-Shapiro & Shafarevich (1971), quintics in $\mathbb{P}^4$ (see Voisin 1999a and cubics in $\mathbb{P}^5$ Voisin 1986).

Let $\pi : \mathcal{Y}' \to B'$ be the universal family of smooth hypersurfaces of degree d in $\mathbb{P}^n$ having no non-trivial automorphism. Let B' denote the quotient by $\mathrm{Gl}(n+1)$ of the open set of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ parametrising smooth hypersurfaces with no non-trivial automorphism. The tangent space of B' at a point f can be naturally identified with R^d_f (see remark 6.16).

On B', we have the global period map

$$\mathcal{P} : B' \to \mathcal{D}/\Gamma,$$

where $\mathcal{D}$ is the period domain parametrising the Hodge structures whose underlying lattice L is isomorphic to $H^{n-1}(Y_f, \mathbb{Z})_{\mathrm{prim}}$, and that have the same Hodge numbers as $H^{n-1}(Y_f, \mathbb{C})_{\mathrm{prim}}$ for any $f \in B'$. Here, the group Γ is the automorphism group of L.

Remark *It is in fact more reasonable to work as in vI.7.1.2 with the polarised period map, where we include the data of the intersection form on the lattices. In this last case,* Γ *has to be the group of automorphism of* $(L, < \ , \ >)$.

By definition, the global period map associates to f the Hodge structure on $H^{n-1}(Y_f, \mathbb{Z})_{\mathrm{prim}}$, which can be considered as a Hodge structure on L via the choice of an isomorphism

$$H^{n-1}(Y_f, \mathbb{Z})_{\mathrm{prim}} \cong L.$$

We know that the period map is holomorphic. The generic Torelli problem is the question of knowing whether $\mathcal{P}$ is of degree 1 over its image. In other words, given two hypersurfaces Y and Y' of degree d in $\mathbb{P}^n$, with Y generic, such that there exists an isomorphism of (polarised) Hodge structures

$$i : H^{n-1}((Y, \mathbb{Z})_{\mathrm{prim}}, F^{\cdot}) \cong H^{n-1}((Y', \mathbb{Z})_{\mathrm{prim}}, F^{\cdot}),$$

one asks whether Y and Y' are isomorphic. In vI.10.3.1 we showed how to obtain this statement in the case of curves of genus ≥ 5, using arguments from the theory of infinitesimal variations of Hodge structure.

In the case of hypersurfaces, Donagi used this method to prove the following result.

Theorem 6.29 *The generic Torelli theorem holds for hypersurfaces of degree* d *in* $\mathbb{P}^n$*, up to the possible exception of the following cases:*

(i) d *divides* $n + 1$;
(ii) $d = 3, \ n = 3$;
(iii) $d = 4, \ n \equiv 1 \mod 4$;
(iv) $d = 6, \ n \equiv 2 \mod 6$.

Remark 6.30 *Case (ii) is a true exception, since all cubic surfaces have isomorphic Hodge structures (they satisfy* $h^{2,0} = 0$*), whereas the quotient* B' *in this case is 4-dimensional.*

Case (iv) was considered by Cox & Green (1990). It is probable that the infinite series of exceptions in case (i) contains only a finite number of counterexamples to the generic Torelli theorem, maybe none. The first cases which occur, namely cubic curves in $\mathbb{P}^2$, surfaces of degree 4 in $\mathbb{P}^3$, and quintics in $\mathbb{P}^4$, have all been solved already.

Note first that thanks to Macaulay's theorem, the period map is an immersion in all the cases considered. This is known as the infinitesimal Torelli theorem for hypersurfaces.

Indeed, it suffices to check that at each point $f \in B'$ the map $d\mathcal{P}$, which by the results of subsection 5.1.2 is induced by the $\overline{\nabla}_p$ via the adjunction relation

$$d\mathcal{P}_f(u) = \bigoplus \langle \overline{\nabla}_{p,u} \rangle \in \bigoplus\nolimits_p \operatorname{Hom}\left(H^{p,n-1-p}(Y_f)_{\mathrm{van}}, H^{p-1,n-p}(Y_f)_{\mathrm{van}}\right),$$

is injective.

Now, by theorem 6.13 and remark 6.16, this map $d\mathcal{P}_f$ can be identified up to a coefficient with the map given by the product

$$R_f^d \to \bigoplus\nolimits_p \operatorname{Hom}\left(R_f^{(n-p)d-n-1}, R_f^{(n-p+1)d-n-1}\right).$$

By corollary 6.20, this map is injective if there exists $p \geq 0$ such that

$$(n-p)d - n - 1 \geq 0, \quad (n-p+1)d - n - 1 \leq N = (d-2)(n+1),$$

which only excludes cubic surfaces and quadratic hypersurfaces of any dimension. The latter have in fact no moduli, so we are left with the cubic surfaces, which is case (ii) above.

As $\mathcal{P}$ is an immersion, it suffices to see that if U and U' are two open sets of B' and $j : U \to U'$ is an isomorphism such that there exists an isomorphism of variations of Hodge structure

$$J : (R^{n-1}\pi_*\mathbb{Z}_{\mathrm{prim}}, F^{\cdot}\mathcal{H}^{n-1}) \cong j^*(R^{n-1}\pi_*\mathbb{Z}_{\mathrm{prim}}, F^{\cdot}\mathcal{H}^{n-1}),$$

then $U = U',\ j = \mathrm{Id},\ J = \mathrm{Id}$.

But at each point $f \in U$, setting $f' = j(f)$, such an isomorphism induces an isomorphism of infinitesimal variations of Hodge structure

$$\begin{array}{ccc} T_{U,f} & \longrightarrow & \bigoplus_p \operatorname{Hom}(H^{p,n-1-p}(Y_f)_{\mathrm{prim}}, H^{p-1,n-p}(Y_f)_{\mathrm{prim}}) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle J} \\ T_{U',f'} & \longrightarrow & \bigoplus_p \operatorname{Hom}(H^{p,n-1-p}(Y_{f'})_{\mathrm{prim}}, H^{p-1,n-p}(Y_{f'})_{\mathrm{prim}}), \end{array} \tag{6.19}$$

i.e. by theorem 6.13 a commutative diagram

$$\begin{tikzcd} R_f^d \arrow[r] \arrow[d] & \bigoplus_p \mathrm{Hom}\left(R_f^{(n-p)d-n-1}, R_f^{(n-p+1)d-n-1}\right) \arrow[d] \\ R_{f'}^d \arrow[r] & \bigoplus_p \mathrm{Hom}\left(R_{f'}^{(n-p)d-n-1}, R_{f'}^{(n-p+1)d-n-1}\right) \end{tikzcd} \tag{6.20}$$

whose vertical arrows are isomorphisms. It thus suffices to prove that the existence of a diagram as above implies that Y_f and Y'_f are isomorphic.

For this, Donagi uses the following lemma.

Lemma 6.31 *Assume that f and f' are homogeneous polynomials of degree d in $n+1$ variables, and that there exists an automorphism $g : S \to S$ of the graded ring S such that $g(J_f) = J_{f'}$. Then there exists an automorphism g' of S such that $g'(f) = f'$.*

Remark 6.32 *One cannot always take $g' = g$, even up to homothety. The simplest counterexample is given by $f = \sum_i X_i^d$, $f' = \sum_i \alpha_i X_i^d$, where the α_i are arbitrary coefficients. Clearly, these polynomials have the same Jacobian ideal, so that we can take $g = \mathrm{Id}$. However, these polynomials are not proportional.*

Proof of lemma 6.31 By replacing f by $g(f)$, we may of course assume that $g = \mathrm{Id}$. Then we know that f and f' have the same associated Jacobian ideal J. But then clearly $f_t = tf + (1-t)f'$ also has the same Jacobian ideal $J_t = J$. Now, $\frac{d}{dt}(f_t) = f - f' \in J = J_t$. As J_t is the tangent space at f_t to the orbit of f_t under the action of $\mathrm{Gl}(n+1)$, we conclude that the trajectory $t \mapsto f_t$ in B projects to a constant in $B' = B/\mathrm{Gl}(n+1)$. Thus, f and f' have the same projection in B', i.e. they are conjugate under the action of $\mathrm{Gl}(n+1)$. □

Donagi's idea was to deduce from the diagram (6.20) a commutative diagram of morphisms of rings

$$\begin{tikzcd} g : S \arrow[r] \arrow[d] & S \arrow[d] \\ R_f \arrow[r] & R_{f'}, \end{tikzcd} \tag{6.21}$$

in which the lower isomorphism is equal to j_* in degree d and is induced by J in the degrees $pd - n - 1$, while the vertical arrows are the natural projections.

This then makes it possible to apply lemma 6.31 to conclude that Y_f and $Y_{f'}$ are isomorphic.

The symmetriser lemma is used to extend the commutative diagram (6.20) as follows. Let k be the smallest non-zero integer of the form $pd - n - 1$. If d does not divide $n+1$, then k is strictly less than d. We know by the symmetriser lemma that given the multiplication

$$R_f^d \times R_f^k \to R_f^{k+d},$$

the set

$$T_f = \left\{\phi \in \mathrm{Hom}\left(R_f^k, R_f^d\right) \mid A\phi(B) = B\phi(A) \in R_f^{k+d}, \ \ \forall A,\ B \in R_f^k\right\}$$

can be identified with R_f^{d-k}, and the natural map

$$T_f \times R_f^k \to R_f^d$$

can be identified with the product. Thus, we have constructed a new piece of the Jacobian ring, i.e. the diagram (6.20) also gives a commutative diagram

$$\begin{tikzcd} R_f^{d-k} \arrow[r] \arrow[d] & \mathrm{Hom}\left(R_f^k, R_f^d\right) \arrow[d, "J"] \\ R_{f'}^{d-k} \arrow[r] & \mathrm{Hom}\left(R_{f'}^k, R_{f'}^d\right), \end{tikzcd}$$

where the horizontal arrows are given by multiplication. Iterating this procedure, we easily see that starting from the diagram (6.20), we can construct in this manner a ring isomorphism

$$R'_f \to R'_{f'},$$

where the index $'$ means that we take the subring consisting of the elements divisible by $\delta = \mathrm{GCD}(d, n+1)$. As we know that $\delta < d$ since d does not divide $n+1$, the components of small degree of R'_f are isomorphic to the corresponding components of the ring S, and we can then show that apart from the exceptions listed above, this is sufficient to construct the diagram (6.21). For example, when $\delta = 1$, the symmetriser lemma gives an isomorphism between the Jacobian rings of f and f', and the degree 1 part gives an isomorphism $g_1 : S^1 = R_f^1 \to S^1 = R_{f'}^1$. For $g : S \to S$, we then take the morphism induced by g_1. □

Donagi's theorem was generalised by Green (1984c) to families of sufficiently ample hypersurfaces of a smooth projective variety.

Exercises

1. In this problem, we propose to construct a smooth projective variety X of dimension 3, and a class $0 \neq \alpha \in H^3(X, \mathbb{Q}) \cap F^1H^3(X)$ which is not contained in

$$j_*H_3(Y, \mathbb{Q}) \subset H_3(X, \mathbb{Q}) \cong H^3(X, \mathbb{Q})$$

for any hypersurface $Y \overset{j}{\hookrightarrow} X$. This is thus a counterexample to the naive Hodge conjecture (see vI.11.3.2). The starting point is the observation that for every hypersurface Y, $j_*H_3(Y, \mathbb{Q}) \subset H^3(X, \mathbb{Q})$ is a Hodge substructure of $H^3(X, \mathbb{Q})$ contained in $F^1H^3(X, \mathbb{Q})$. Hence it suffices to construct such a pair (X, α) which moreover satisfies the property:

There exists no non-zero Hodge substructure L_t of $H^3(X, \mathbb{Q})$ contained in $F^1H^3(X, \mathbb{Q})$.

Such a Hodge structure is given by a vector subspace $L_{\mathbb{Q}}$ of $H^3(X, \mathbb{Q})$ contained in $F^1H^3(X)$, such that we have an induced decomposition

$$L_{\mathbb{C}} = L^{2,1} \oplus L^{1,2},$$

$$L^{2,1} := L_{\mathbb{C}} \cap F^2H^3(X), \quad L^{1,2} = \overline{L^{2,1}} \subset F^1H^3(X).$$

The varieties we will consider are smooth quintic hypersurfaces of $\mathbb{P}^4$. Such a hypersurface X satisfies $K_X \cong \mathcal{O}_X$ by the adjunction formula, so that $h^{0,3}(X) = 1$. Choose a connected and simply connected open set

$$U \subset H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)) = S^5$$

containing 0 and parametrising smooth hypersurfaces. For $\alpha \in H^3(X_0, \mathbb{Q})$, let $\alpha_t \in H^3(X_t, \mathbb{Q})$, $t \in U$ denote the class deduced from α by the canonical isomorphism $H^3(X_0, \mathbb{Q}) \cong H^3(X_t, \mathbb{Q})$ given by the simple connectedness of U.

(a) Adapting the proofs of propositions 5.14 and 5.20, show that for $0 \neq \alpha \in H^3(X_0, \mathbb{Q})$, the sets

$$U_\alpha := \{t \in H^3(X_0, \mathbb{Q}) \mid \alpha_t \in F^1H^3(X_t, \mathbb{Q})\}$$

are hypersurfaces of U, and the union of the U_α for $\alpha \neq 0$ is dense in U.

Fix α such that $U_\alpha \neq \emptyset$. We propose to show that the generic point t of U_α is such that there exists no non-zero Hodge substructure L of $H^3(X_t, \mathbb{Q})$ contained in $F^1H^3(X_t)$. Assume, on the contrary, that such a

https://doi.org/10.1017/CBO9780511615177.007 Published online by Cambridge University Press

Hodge substructure L_t exists, and let $L_t \subset H^3(X_t, \mathbb{Q})$ be its underlying $\mathbb{Q}$-vector space. By a countability argument, we may assume that L_t does not depend on t (or more precisely, that it is locally constant).

(b) Show that for $t \in U_\alpha$ and $u \in T_{U_\alpha,t}$, and for every

$$\lambda \in L_t^{1,2} \cong L_{\mathbb{C}}/L_t^{2,1} \subset F^1/F^2H^3(X_t),$$

we have

$$\overline{\nabla}\lambda(u) = 0 \text{ in } H^{0,3}(X_t).$$

(c) Deduce from theorems 6.13 and 6.19, together with the fact that $T_{U_\alpha,t}$ is a hyperplane of S^5, that $\dim L_t^{1,2} \leq 1$.

(d) Assuming that L is non-zero, let λ be a generator of $L_t^{2,1}$. Show that

$$\overline{\nabla}(\lambda)(u) \in L_t^{1,2} \subset F^1/F^2H^3(X_t)$$

for $u \in T_{U_\alpha,t}$.

(e) Let H be a hyperplane of S^5 with no base point (i.e. such that there exists no point of $\mathbb{P}^4$ on which all the elements of H vanish). Show that H generates S^6, i.e. that

$$S^1 \cdot H = S^6.$$

Similarly, let $K \subset S^5$ be a subspace of codimension 2 with no base point. Show that $K \cdot S^2 = S^7$.

(f) Now let H be the hyperplane of S^5 given by $T_{U_\alpha,t}$. Show that H has no base point, and that $\overline{\nabla}(\lambda)(H)$ is contained in $L^{1,2}$. Let $K \subset S^5$ be the subspace of H defined by

$$K = \mathrm{Ker}\left(\overline{\nabla}(\lambda) : H \to L^{1,2}\right).$$

Show that K is of codimension 2, with no base point.

(g) Let $P \in R^5_{f_t}$ be the polynomial representing λ under the isomorphism of corollary 6.12. Show that

$$P \cdot K = 0 \text{ in } R^{10}_{f_t}.$$

(h) Deduce from (e) and (g) that $P \cdot R^7_{f_t} = 0$ in $R^{12}_{f_t}$. Deduce from theorem 6.19 that $P = 0$, so that $L_t = 0$. This contradicts the assumption that L_t is non-zero.

2. *Components of small codimension of the Noether–Lefschetz locus.* In this problem, we propose to prove the theorem of Voisin (1988) and Green (1989a) in the case where $d = 5$.

Let U denote a connected and simply connected open subset of the open set of $S^5 := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$ parametrising smooth surfaces. Let $0 \in U$, and let $0 \neq \lambda \in H^2(S_0, \mathbb{Z})_{\mathrm{prim}}$. Consider the component U_λ of the Noether–Lefschetz locus given by

$$U_\lambda = \{t \in U \mid \lambda_t \in F^1H^2(S_t)\}.$$

Assume that $0 \in U_\lambda$.

(a) Let $P_\lambda \in R^6_{f_0}$ be the polynomial representing $\lambda \in H^{1,1}(S)_{\mathrm{prim}}$ under the isomorphism of corollary 6.12. Using theorem 6.13, show that

$$T_{U_\lambda,0} = \mathrm{Ker}\left(\mu^5_{P_\lambda} : S^5 \to R^{11}_{f_0}\right)$$

where $\mu^i_{P_\lambda}$ denotes multiplication by P_λ, mapping S^i to $R^{6+i}_{f_0}$.

(b) Let $A := \mathrm{Ker}\,(\mu^1_{P_\lambda} : S^1 \to R^7_{f_0})$. By Macaulay's theorem 6.19, A can be identified with the dual of

$$\mathrm{Coker}\left(\mu^5_{P_\lambda} : S^5 \to R^{11}_{f_0}\right).$$

Show that $A \cdot R^5_{f_0} \subset P_\lambda^\perp$, where $\perp$ is relative to the intersection form on $R^6_{f_0}$ given by Macaulay's theorem.

(c) Show that if $B \subset S^1$ is a vector subspace of rank 3, then $B \cdot R^5_{f_0} = R^6_{f_0}$. Deduce that codim $A \geq 2$, hence that codim $U_\lambda \geq 2$.

Assume from now on that codim $A = 2$. By (b), this is equivalent to assuming that codim $T_{U_\lambda,0} = 2$.

(d) Let $\Delta \subset \mathbb{P}^3$ be the line defined by $A \subset S^1 = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Deduce from (b) that $I_\Delta(6) + J^6_{f_0} \neq S^6$, where $I_\Delta(6) \subset S^6$ is the set of polynomials of degree 6 vanishing on Δ.

(e) Deduce that rank $J^4_{f_0|\Delta} = 2$. (Use the fact that $J^4_{f_0|\Delta} \subset H^0(\mathcal{O}_\Delta(4)$ has no base point.) Show that $I_\Delta(6) + J^6_{f_0}$ is a hyperplane of S^6.

We have thus proved the following result:

Every component U_λ of the Noether–Lefschetz locus is of codimension at least 2, and if U_λ is of codimension equal to 2, then for every point $0 \in U_\lambda$, there exists a line Δ satisfying the following properties:

(i) rank $J^4_{f_0|\Delta} = 2$.

(ii) $I_\Delta(5) \subset T_{U_\lambda}$.

(iii) *The hyperplane* $I_\Delta(6) + J^6_{f_0}$ modulo $J^6_{f_0}$ *is equal to the hyperplane* $P_\lambda^\perp \subset R^6_{f_0}$.

(f) Let $G = G(2,4)$ be the Grassmannian of lines of $\mathbb{P}^3$. Let $Z \subset G \times U$ be the algebraic subset defined by

$$Z = \left\{(\Delta, f) \mid \operatorname{rank} J^4_{f|\Delta} = 2\right\}.$$

Show that the projection of $T_{Z,(\Delta,f)}$ onto $T_{U,f}$ does not contain $I_\Delta(5)$ if Δ is not contained in the surface of equation f.

(g) Deduce from (f) that the line Δ of (e) must be contained in the surface S_0. Deduce from condition (iii) of (e) that λ is proportional to $h - 5[\Delta]$, $h = c_1(\mathcal{O}_{S_0}(1))$. (Hint: Use proposition 5.19 to show that the class $h - 5[\Delta]$ also satisfies condition (iii)).

We have thus proved the following result:

Every component U_λ of the Noether–Lefschetz locus for surfaces of degree 5 in $\mathbb{P}^3$ is of codimension at least 2, and the only component of codimension 2 is the family of surfaces containing a line Δ.