The n/d conjecture for nonresonant hyperplane arrangements

1) Background
   - Definition of b-function & examples
   - The statement of the n/d conjecture
   - Existing results and our main theorem
2) Walther's sufficient condition
3) Proof of our result
   - Strategy : Algebraic de Rham theorem

Appendix : i) Proof of Walther's theorem

ii) The STV log resolution

iii) Algebraic de Rham theorem

iv) Computation of the residues

v) Indecomposability lemma

1. Background

Def : $f \in R = \mathbb{C}[x_1, \cdots, x_n]$ non-constant, $D = R[\partial_1, \cdots, \partial_n]$ Weyl algebra

The global Bernstein-Sato polynomial (b-function) $b_f(s)$ is the minimal polynomial satisfying that $\exists\, P(s) \in D[s]$ s.t.

$$P(s) \cdot f^{s+1} = b_f(s) \cdot f^s$$

Existence:
Global : Bernstein 1972
Local : Björk 1974

Rmk : If we replace $\mathbb{C}[x_1, \cdots, x_n]$ by $\mathcal{O}_{\mathbb{C}^n, x}$ for some $x \in \mathbb{C}^n$, we obtain the local one $b_{f,x}(s)$. In general, $b_f(s) = \operatorname{lcm}_{x \in \mathbb{C}^n} b_{f,x}(s)$. RHS is well-defined since $b_{f,x}(s) = s+1$ when f is smooth at x. Furthermore, if f is quasi-homogeneous, then $b_f(s) = b_{f,0}(s)$

All roots of $b_{f,0}(s)$ are negative rational numbers, (Isolated : Malgrange 1975; General : Kashiwara 1976) and they are widely connected with other objects :

① $\{\exp(2\pi i \alpha) \mid b_{f,0}(\alpha) = 0\} = \{$monodromy eigenvalues of some pt in $f^{-1}(0)\}$ Malgrange 1983

eigenvalues of monodromy on cohomological Milnor fibration

② $\{$poles of $\underbrace{Z_{f,\varphi}(s)}_{= \int_{\mathbb{C}^n} |f(x)|^{2s} \varphi(x)\, dx}\} \subset \{\alpha - k \mid k \in \mathbb{Z}_{\geq 0},\ b_f(\alpha) = 0\}$ Igusa 2000

$Z_{f,\varphi}(s) = \int_{\mathbb{C}^n} |f(x)|^{2s} \varphi(x)\, dx$, the complex Zeta function

Moreover, in 1992, Denef and Loeser conjectured that the poles of the topological zeta function are roots of the b-function. This mysterious problem, which is known as the topological monodromy conjecture, remains widely open in general.

Conj : (Topological monodromy conjecture)

$f \in \mathbb{C}[x_1, \cdots, x_n]$, $\mu : Y \longrightarrow \mathbb{C}^n$ log resolution of $f^{-1}(0)$

E_i component of $(f \circ \mu)^{-1}(0)$, $a_i = \operatorname{ord}_{E_i} f \circ \mu$, $k_i = \operatorname{ord}_{E_i}(\operatorname{Jac}(\mu))$, $E_I^\circ = \bigcap_{i \in I} E_i - \left(\bigcup_{j \notin I} E_j\right)$

$$Z^{top}_{f,x}(s) := \sum_{I \subset S} \chi(E_I^\circ \cap \mu^{-1}(x)) \cdot \prod_{i \in I} \frac{1}{a_i s + k_i + 1}, \quad x \in f^{-1}(0)$$

Then any pole of $Z^{top}_{f,x}(s)$ is a root of $b_{f,x}(s)$

Though the roots of $b_f(s)$ are important, it's very difficult to compute them

The followings are some of the few examples.

1) $f(x) = x^a$, $b_f(s) = (s+\frac{1}{a})(s+\frac{2}{a})\cdots(s+1)$

2) $f(x_1,\cdots,x_n) = \prod_{i=1}^{n} x_i^{a_i}$, $b_f(s) = \prod_{i=1}^{n}\prod_{j=1}^{a_i}(s+\frac{j}{a_i})$

Rmk: Product formula: $f(x_1,\cdots,x_n) = g(x_1,\cdots,x_i)\,h(x_{i+1},\cdots,x_n) \Rightarrow b_f = b_g \cdot b_h$ Shi, Zuo 2024; Lee 2024

3) f is a quasi-homogeneous polynomial with an isolated singularity at 0.

$b_f(s)$ can be calculated via the algebraic Gauss-Manin connection Malgrange 1975

e.g. $f(x_1,\cdots,x_n) = \sum_{i=1}^{n} x_i^2$, $b_f(s) = (s+1)(s+\frac{n}{2})$

4) $f = \prod_{j=1}^{d} L_j$ generic hyperplane arrangement of degree $d > n \geq 2$

$b_f(s) = \prod_{i=0}^{2d-n-2}(s+\frac{n+i}{d}) \cdot (s+1)^{n-1}$ Walther 2005. form ; Saito 2016. exponent of $s+1$

The n/d conjecture is about whether $-\frac{n}{d}$ is a root of $b_f(s)$ for a hyperplane arrangement $f = \prod_{j=1}^{r} L_j^{a_j}$ $(a_j \in \mathbb{Z}_{>0})$ of degree d, where L_j are pairwisely non-collinear linear functionals in $\mathbb{C}^n$.

We have seen this holds for generic cases but fails for monomials due to product formula.

So the definition of indecomposability naturally arises:

Def: A subset B in a linear space V is called indecomposable if:

$\forall$ nontrivial decomposition $V = W_1 \oplus W_2$, $B \not\subset W_1 \cup W_2$

Rmk: We call a hyp. arr f indecomposable if $\mathcal{A} = \{L_j\}$ is indecomposable in $(\mathbb{C}^n)^\vee$

i.e. Under any coordinate transformation, f CANNOT be factorized as:

$$f(x_1,\cdots,x_n) = g(x_1,\cdots,x_i)\cdot h(x_{i+1},\cdots,x_n) \quad \text{for some } 1 \leq i < n$$

Conj: (n/d conjecture, Budur, Mustaţă and Teitler, 2011)

f : an indecomposable central hyperplane arrangement in $\mathbb{C}^n$

$d = \deg f$

Then $-\frac{n}{d}$ is a root of $b_f(s)$

Rmk: Budur, Mustaţă and Teitler proved that if the n/d conjecture holds, then the topological monodromy conjecture holds for all hyperplane arrangements.

However, Veys gives an example s.t. $-\frac{n}{d}$ is not a pole of $Z^{top}_{f,0}(s)$. So it must not be necessary

Existing results: Walther 2005 generic position

A sufficient condition: non-vanishing of certain class in the cohomology of the Milnor fiber

Budur, Saito, Yuzvinsky 2011 certain reduced cases

A standard method to compute the cohomology of the Milnor fiber of a hyp. arr.:

Algebraic de Rham theorem

Walther 2017 Condition (A_s) (including tame, free, $n \le 3$ ...)

Algebraic. $\mathrm{Ann}_D(f^s)$ is generated by derivations

Shi, Zuo 2024 generic multiplicity

Variation of Archimedean Zeta function

We use Walther's sufficient condition and a framework similar to [BSY11] to prove the following results:

Thm (X. Yu 2025)

$f = \prod_{j=1}^{r} L_j^{a_j}$ indecomposable hyp. arr, $A = \{L_1, \cdots, L_r\}$

$\mathcal{L} = \{ W \subsetneqq (\mathbb{C}^n)^\vee \mid A \cap W \text{ is indecomposable in } W \}$ → Dense edges, related to a canonical log resolution of $\{f=0\}$

If $\forall W \in \mathcal{L}$, $\dim W - \frac{n}{d} \sum_{L_j \in W} a_j \notin \mathbb{Z}_{>0}$ (R)

then $b_f(-\frac{n}{d}) = 0$

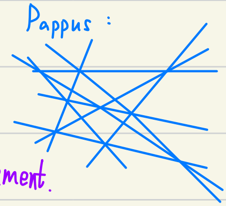
Rmk: 1) (R) holds naturally when $\dim W = 1$ $\Rightarrow$ n/d conj holds for $n = 2$

2) (R) is a common condition in research of hyperplane arrangements:

It's a sufficient condition for the cohomology of a local system on the complement of the hyp. arr. only living in the top degree, which also can be seen in our proof.

Note that not all local systems have this property.

Pappus:



e.g. $xyz(x-y)(y-z)(x-y-z)(2x+y+z)(x+2y+3z)(-2x+5y-z)$

So our strategy must fail for some "resonant" hyperplane arrangement.

(3) (R) gives an explicit description of the remaining cases, which is compatible with a result of Budur, Shi and Zuo in 2024. It says that for fixed A, $\exists$ a non-zero product Θ of linear factors in r variables s.t. for any $\underline{a} \in \mathbb{Z}^r - \{\Theta = 0\}$, $-\frac{n}{\sum_{j=1}^{r} a_j}$ is a pole of the motivic zeta function.

$$\begin{tikzcd}
\text{Motivic} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] \arrow[rr, bend right=60, "\text{for generic multiplicity}"'] & \text{topo.} & n/d \arrow[l, Rightarrow] \\
 & p\text{-adic} &
\end{tikzcd}$$

2. Walther's sufficient condition

Setting: $f \in \mathbb{C}[x_1, \cdots, x_n]$ a homogeneous polynomial of degree $d > 0$

Affine Milnor fibration: $f : \mathbb{C}^n - f^{-1}(0) \longrightarrow \mathbb{C} - \{0\}$

Rmk: It is fiber diffeomorphic to the original Milnor fibration.

Milnor fiber $\quad F = f^{-1}(1)$

Thm: (See Walther 2005, Thm 4.12)

Let $f \in \mathbb{C}[x_1, \cdots, x_n]$ be a homogeneous polynomial of degree $d > 0$

Denote by $\omega_0 = \frac{1}{d} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

If ω_0 defines a nonzero cohomology class in $H^{n-1}(F, \mathbb{C})$,

then $b_f(-\frac{n}{d}) = 0$

We can go further to restrict $[\omega_0]$ in a smaller cohomology group

Denote by $D = \{f = 0\}$ the reduced divisor defined by f on $\mathbb{P}^{n-1}$. $U = \mathbb{P}^{n-1} - D$.

Consider the automorphism $\rho : F \longrightarrow F$, $(x_1, \cdots, x_n) \longmapsto \underbrace{\zeta}_{= e^{\frac{2\pi i}{d}}} (x_1, \cdots, x_n)$.

It induces a covering map $p : F \longrightarrow U$, $\quad (x_1, \cdots, x_n) \longmapsto [x_1 : \cdots : x_n]$

ρ induces automorphisms on cohomologies and sheaves, which are compatible:

$$\begin{tikzcd}
H^q(F, \mathbb{C}) \arrow[r, phantom, "\simeq"] & H^q(U, p_* \underline{\mathbb{C}}_F) \arrow[rr, leftrightarrow, bend right, "\text{Riemann-Hilbert}"'] & & (p_* \mathcal{O}_F, d_F) \\
H^q(F, \mathbb{C})_{\zeta^k} \arrow[u, hook] \arrow[r, phantom, "\simeq"] & H^q(U, (p_* \underline{\mathbb{C}}_F)_{\zeta^k}) \arrow[u, hook] \arrow[rr, leftrightarrow, bend right, "\text{Riemann-Hilbert}"'] & & \left((p_* \mathcal{O}_F)_{\zeta^k}, d_F|_{(p_* \mathcal{O}_F)_{\zeta^k}} \right) \arrow[u, hook]
\end{tikzcd}$$

differential on $\mathcal{O}_F$

$H^q(F, \mathbb{C})_{\zeta^k}$ $=$ means eigenspace of ρ^*

In particular, since $\rho^* \omega_0 = \zeta^n \omega_0$, it suffices to compute the cohomology of

$\mathbb{L} = (p_* \underline{\mathbb{C}}_F)_{\zeta^n} \xleftrightarrow{\text{R-H}} \left(\mathcal{V} = (p_* \mathcal{O}_F)_{\zeta^n} = \mathcal{O}_U(n), \nabla = d_F|_{\mathcal{V}} : \text{"} g \cdot f^{-\frac{n}{d}} \longmapsto d(g \cdot f^{-\frac{n}{d}}) \text{"} \right)$.

3. Proof of our result

We will use the algebraic de Rham Theorem to describe $H^{n-1}(U, \mathbb{L})$.

Recall our setting is :

$f = \prod L_j^{a_j}$ indecomposable hyp. arr. $\rightsquigarrow$ reduced one $A = \{L_j\} \subset (\mathbb{C}^n)^\vee$

Want to know whether $[\omega_0] \neq 0$ in $H^{n-1}(U, \mathbb{L})$

given by Schechtman, Terao and Varchenko, 1995

Since f is a hyp. arr., there is a canonical log resolution $\pi : Y \longrightarrow \mathbb{P}^{n-1}$ of D,

s.t. π is a blowing-up with centers in D s.t. $E = \pi^{-1}D$ is s.n.c.

and $\mathcal{L} = \{$ Dense edges $\}$ $\xleftrightarrow{1:1}$ $\{$irreducible components of $E\}$

$$W \longleftrightarrow E_W$$

Recall that $\mathcal{V} = \mathcal{O}_U(n)$ So a locally free extension of $\mathcal{V}$ on Y has the form

$$\mathcal{V}_\mu = \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(n) \otimes \mathcal{O}_Y\Big(-\sum_{W \in \mathcal{L}} \mu(W) E_W\Big) \quad , \text{ where } \mu : \mathcal{L} \to \mathbb{Z},$$

Furthermore, by local computation, we always have $\nabla(\mathcal{V}_\mu) \subset \Omega^1_Y(\log E) \otimes \mathcal{V}_\mu$,

i.e. $(\mathcal{V}_\mu, \nabla_\mu = \nabla|_{\mathcal{V}_\mu} : \mathcal{V}_\mu \longrightarrow \Omega^1_Y(\log E) \otimes \mathcal{V}_\mu)$ is a regular extension of $(\mathcal{V}, \nabla)$,

and the residue of ∇_μ along E_W $(W \in \mathcal{L})$ is exactly $\mu(W) - \frac{n}{d} \sum_{L_j \in W} a_j$

Thm (A corollary of the algebraic de Rham theorem) Deligne 1970

If none of the residues $\mu(W) - \frac{n}{d} \sum_{L_j \in W} a_j$ are positive integers, the canonical map

$$\mathbb{H}^p(Y, \Omega^\bullet_Y(\log E) \otimes \mathcal{V}_\mu) \longrightarrow H^p(U, \mathbb{L})$$

is an isomorphism.

Now we have the following sequence :

$$H^0(Y, \Omega^{n-1}_Y(\log E) \otimes \mathcal{V}_\mu) \xrightarrow{\text{edge}} \mathbb{H}^{n-1}(Y, \Omega^\bullet_Y(\log E) \otimes \mathcal{V}_\mu) \xrightarrow{\text{alg. de Rham}} H^{n-1}(U, \mathbb{L})$$

Our goal is to find μ s.t

$$w_0 \overset{①}{\in} H^0(Y, \Omega_Y^{n-1}(\log E)\otimes \mathcal{V}_\mu) \overset{②}{\hookrightarrow} \mathbb{H}^{n-1}(Y, \Omega_Y^{\bullet}(\log E)\otimes \mathcal{V}_\mu) \overset{③}{\xrightarrow{\sim}} H^{n-1}(U, \mathbb{L})$$

① w_0 defines a nonzero section in $H^0(Y, \Omega_Y^{n-1}(\log E)\otimes \mathcal{V}_\mu)$

Note that $\mathbb{C} w_0 = H^0(\mathbb{P}^{n-1}, \Omega^{n-1}_{\mathbb{P}^{n-1}}(n)) = H^0(Y, \Omega_Y^{n-1}(\log \pi^{-1} D)\otimes \mathcal{V}_{\dim})$

So it suffices to take $\mu \leq \dim$ $\dim : \mathcal{L} \to \mathbb{Z}, \ W \mapsto \dim W$

② $H^p(Y, \Omega_Y^q(\log E)\otimes \mathcal{V}_\mu) = 0, \ \forall \ p+q = n-2$ then the edge morphism is injective.

Our main tool is the following vanishing theorem :

Thm (Esnault & Viehweg 1992)

Y projective, $E = \sum_{j=1}^{l} E_j$ SNC, $Y - E$ affine

(*) $\mathcal{V}$ invertible s.t. $\mathcal{V}^N = \mathcal{O}_Y(\sum_{j=1}^{l} c_j E_j)$ for some integers $0 < c_1, \cdots, c_l < N$

Then $H^p(Y, \Omega_Y^q(\log E)\otimes \mathcal{V}^{-1}) = 0$ holds for any $p+q \neq \dim Y$

The dual version :

$H^p(Y, \Omega_Y^q(\log E)\otimes \mathcal{V}\otimes \mathcal{O}_Y(-E)) = 0$ holds for any $p+q \neq \dim Y$

In our case : $\mathcal{V} = \mathcal{V}_\mu \otimes \mathcal{O}_Y(E) = \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(n) \otimes \mathcal{O}_Y(\sum_{W\in\mathcal{L}} (1-\mu(W)) E_W)$

So (*) $\Leftrightarrow \exists \ \varepsilon_1, \cdots, \varepsilon_r \in \mathbb{Q}$, $\sum_{j=1}^{r} \varepsilon_j = n$ s.t.

$$\forall \ W \in \mathcal{L}, \ 0 < \mu(W) - \sum_{L_j \in W} \varepsilon_j < 1$$

Conversely, indecomposability enables us to find $\varepsilon_1, \cdots, \varepsilon_r \in \mathbb{Q}$

s.t. $\sum_{j=1}^{r} \varepsilon_j = n$, and $\forall \ W \in \mathcal{L}$, $\sum_{L_j \in W} \varepsilon_j < \dim W$ Budur, Shi, Zuo 2024

Then choose $\mu(W) = \lceil \sum_{L_j \in W} \varepsilon_j \rceil$ disturb ε_j to make $\sum_{L_j\in W} \varepsilon_j \notin \mathbb{Z}$

We have both ① and ② hold

Now for ③ we only need that all residues of $(\mathcal{V}_\mu, \nabla_\mu)$ are not positive integers

i.e. $\mu(W) - \frac{n}{d} \sum_{L_j \in W} a_j \notin \mathbb{Z}_{>0}$, $\forall W \in \mathcal{L}$ then alg. de Rham gives isomorphisms.

Since $\mu \leq \dim$, this can be guaranteed by the nonresonant condition:

$$\forall W \in \mathcal{L}, \quad \dim W - \frac{n}{d} \sum_{L_j \in W} a_j \notin \mathbb{Z}_{>0} \qquad (R)$$

So the n/d conjecture holds for nonresonant hyperplane arrangements.

Rmk: Previous work chose $\mathcal{V}_\mu$ s.t. $H^p(Y, \Omega_Y^q(\log E) \otimes \mathcal{V}) = 0$ for $p>0$.

So they still need to prove $\Gamma(Y, \Omega_Y^{n-2}(\log E) \otimes \mathcal{V}_\mu) \longrightarrow \Gamma(Y, \Omega_Y^{n-1}(E) \otimes \mathcal{V}_\mu)$ doesn't kill w_0.

Appendix 1 : Proof of Walther's theorem

Notation :

f : homogeneous polynomial of degree d with n variables.

$R = \mathbb{C}\{x_1, \cdots, x_n\}$, $D = R\langle \partial_1, \cdots, \partial_n \rangle$, $\mathcal{O} = \mathbb{C}\{t\}$, $K = \mathrm{Frac}(\mathcal{O})$

$M = D[s]f^s$, $\mathcal{O} \curvearrowright M$: $t \cdot P(s) f^s = P(s+1) f^{s+1}$

Then $b_{f,0}(s)$ is the minimal polynomial of $s : M/tM \longrightarrow M/tM$

$DR[M]$: $0 \longrightarrow M \longrightarrow M \otimes_R \Omega^1 \longrightarrow \cdots \longrightarrow M \otimes_R \Omega^n \longrightarrow 0$

$$d(m \otimes w) = m \otimes dw + \sum_{i=1}^n \partial_i m \otimes (dx_i \wedge w)$$

Relative differential form $\Omega_f^p = \Omega_X^p / df \wedge \Omega_X^{p-1}$

Relative de Rham $\mathcal{H}^p = \mathcal{H}^p(f_* \Omega_f^\bullet)$

$$\mathcal{H}^p|_{T'} \simeq R^p f_* \mathbb{C}_{X'} \otimes_{\mathbb{C}_{T'}} \mathcal{O}_{T'} \quad \text{locally free}$$

The whole picture :

$$[f^s \otimes dx] \neq 0 \xRightarrow{\text{Nontrivial ! homogeneous}} b_f\left(-\frac{n}{d}\right) = 0 \qquad s \cdot f^s = -\frac{n}{d} f^s + \frac{1}{d}\sum_{i=1}^n \partial_i (x_i f^s)$$

$$\begin{tikzcd}
{[f^s \otimes dx]} \arrow[d, phantom, sloped, "\in"] & \\
H^n(DR[M]) \arrow[d] & H^{n-1}(\Omega^\bullet_{f,0}) \arrow[l] \arrow[d] \\
H^n(DR[M]) \otimes_{\mathcal{O}} K & H^{n-1}(\Omega^\bullet_{f,0}) \otimes_{\mathcal{O}} K \arrow[l, "\simeq"']
\end{tikzcd}$$

quasi-homogeneous

$$H^n(DR[M]) \otimes_{\mathcal{O}} K \xleftarrow{\simeq} H^{n-1}(\Omega^\bullet_{f,0}) \otimes_{\mathcal{O}} K = \mathcal{H}^{n-1} \otimes_{\mathcal{O}} K$$

chain map. $f^s \otimes \left(\frac{df}{f} \wedge w\right) \longleftarrow [w]$

Malgrange's observation

Need Euler's identity

$$[f^s \otimes dx] \longleftrightarrow \left[w_0 = \frac{1}{d}\sum_{i=1}^n x_i \widehat{dx_i}\right] \neq 0$$

$\Uparrow$ $\mathcal{H}^{n-1}$ is locally free on T'

$[w_0] \neq 0$ in $H^{n-1}(F, \mathbb{C})$

Appendix 2: The STV log resolution

Setting: $\mathcal{A}$ is a reduced hyp. arr. in $\mathbb{P}^{n-1}$

i.e. $\mathcal{A} \subset (\mathbb{C}^n)^\vee$ a finite collection of pairwisely non-collinear functionals

Denote by

$\mathcal{L} = \{ W \subsetneqq (\mathbb{C}^n)^\vee \mid \mathcal{A} \cap W \text{ is indecomposable in } W \}$ dense edges.

$\mathcal{L}_k = \{ W \in \mathcal{L} \mid \dim W = k \}, \quad 1 \le k \le n-1$

$\mathcal{L}_k$ defines a reduced subvariety Z_k of codimension k. $Z_1 = D$

$$\pi : Y = Y_1 \xrightarrow{\tau_2} Y_2 \xrightarrow{\tau_3} \cdots \xrightarrow{\tau_{n-2}} Y_{n-1} \xrightarrow{\tau_{n-1}} Y_{n-1} = \mathbb{P}^{n-1},$$

where $\tau_k : Y_{k-1} \longrightarrow Y_k$ is the blowing-up along the proper transform of Z_k.

Lem: (Schechtman, Terao and Varchenko, 1995)

π is a blowing-up with centers in D s.t. $E = \pi^{-1} D$ is s.n.c.

In particular, $\mathcal{L} \xleftrightarrow{1:1} \{ \text{irreducible components of } E \}$

$$W \longleftrightarrow E_W$$

Appendix 3: Algebraic de Rham theorem

Setting: X = a complete complex analytic variety

D = a reduced divisor on X, $\mathcal{U} = X - D$

$\mathbb{L}$ = a local system on $\mathcal{U} \xleftrightarrow{R\text{-}H} (\mathcal{V}, \nabla)$

Fix a resolution $\pi: Y \longrightarrow X$ of D s.t. $E = \pi^{-1}D$ is a simple normal crossing divisor containing all exceptional divisors. Denote by $j: \mathcal{U} \hookrightarrow Y$ the open immersion.

Then ∇ induces $j_*\mathcal{V} \longrightarrow j_*(\Omega^1_{\mathcal{U}} \otimes \mathcal{V})$, which we still denote by ∇.

Def: If a locally free extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ on Y satisfies that

$$\nabla(\tilde{\mathcal{V}}) \subset \Omega^1_Y(\log E) \otimes \tilde{\mathcal{V}} \subset j_*(\Omega^1_{\mathcal{U}} \otimes \mathcal{V}),$$

then we call $(\tilde{\mathcal{V}}, \tilde{\nabla} = \nabla|_{\tilde{\mathcal{V}}}: \tilde{\mathcal{V}} \longrightarrow \Omega^1_Y(\log E) \otimes \tilde{\mathcal{V}})$ a regular extension of $(\mathcal{V}, \nabla)$

Rmk: By definition $(\tilde{\mathcal{V}}, \tilde{\nabla})$ is a regular connection, so we can define residue maps.

∇ is integrable, so as $\tilde{\nabla}$. So $(\tilde{\mathcal{V}}, \tilde{\nabla})$ defines a log de Rham complex:

$$\tilde{\nabla}: \Omega^p_Y(\log E) \otimes \tilde{\mathcal{V}} \longrightarrow \Omega^{p+1}_Y(\log E) \otimes \tilde{\mathcal{V}}, \quad w \otimes v \longmapsto dw \otimes v + (-1)^p w \wedge \tilde{\nabla} v$$

And there exists a canonical map $\mathbb{H}^*(Y, \Omega^\bullet_Y(\log E) \otimes \tilde{\mathcal{V}}) \longrightarrow H^*(\mathcal{U}, \mathbb{L})$

Thm: (Deligne 1970)

If the residue of $\tilde{\nabla}$ along each component of E does not have any positive integer values, then we have

$$\mathbb{H}^p(Y, \Omega^\bullet_Y(\log E) \otimes \tilde{\mathcal{V}}) \xrightarrow{\sim} H^p(\mathcal{U}, \mathbb{L})$$

Appendix 4 : Computation of residue

Setting: f hyp. arr. of degree d in $\mathbb{P}^{n-1}$

$D = \{f = 0\}$, $\mathcal{U} = \mathbb{P}^{n-1} - D$

$\pi : Y \longrightarrow \mathbb{P}^{n-1}$ the STV log resolution, $j : \mathcal{U} \hookrightarrow Y$

$E = \pi^{-1} D$ whose irr. component $E_W \overset{1:1}{\longleftrightarrow} W \in \mathcal{L} = \{\text{dense edges}\}$

$\mathcal{V} = \mathcal{O}_{\mathcal{U}}(n)$. For $\mu : \mathcal{L} \longrightarrow \mathbb{Z}$, $\mathcal{V}_\mu = \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(n) \otimes \mathcal{O}_Y(-\sum_{W \in \mathcal{L}} \mu(W) E_W) \subset j_* \mathcal{V}$

$\nabla : \mathcal{V} \longrightarrow \Omega^1_{\mathcal{U}} \otimes \mathcal{V}$, " $g \cdot f^{-\frac{n}{d}} \longmapsto d(g \cdot f^{-\frac{n}{d}})$ " $\rightsquigarrow j_* \mathcal{V} \longrightarrow j_* (\Omega^1_{\mathcal{U}} \otimes \mathcal{V})$

More explicitly, for any hyperplane H in $\mathbb{P}^{n-1}$ defined by L, denote by $f_L = \frac{f}{L^d}$

Then $\nabla|_{\mathcal{U}-H} : \mathcal{O}_{\mathcal{U}-H} \longrightarrow \Omega^1_{\mathcal{U}-H}$, $g \longmapsto dg - g \cdot \frac{n}{d} \cdot \frac{df_L}{f_L}$

Lem: $\forall \mu : \mathcal{L} \to \mathbb{Z}$, $\nabla(\mathcal{V}_\mu) \subset \Omega^1_Y(\log E) \otimes \mathcal{V}_\mu$ and $\mathrm{Res}_{E_W} \nabla_\mu = \mu(W) - \frac{n}{d} \sum_{L_j \in W} a_j$

Proof: $\forall y \in Y$, take a small neighbourhood U_y of y.

Assume $E = \{z_1 \cdots z_m = 0\}$ on U_y since E is s.n.c. Suppose E_{W_i} corresponds to $\{z_i = 0\}$.

Choose H to be generic ($\pi^{-1} H \cap U_y = \emptyset$). Then by definition, on U_y:

$\nabla : j_* \mathcal{O}_{U_y - E} \longrightarrow j_* \Omega^1_{U_y - E}$, $g \longmapsto dg - \frac{n}{d} \cdot g \cdot \frac{d(f_L \circ \pi)}{f_L \circ \pi}$ where $j : U_y - E \hookrightarrow U_y$

$$\mathcal{V}_\mu = \underbrace{\pi^* \mathcal{O}_{U_y}(n)}_{\text{trivilized by } (L\circ\pi)^n \text{ as in } \nabla} \otimes \underbrace{\mathcal{O}_{U_y}(-\sum_{W \in \mathcal{L}} \mu(W) E_W)}_{\text{trivilized by } \prod_{i=1}^m z_i^{\mu(W_i)}} \simeq \mathcal{O}_{U_H} \cdot \prod_{i=1}^m z_i^{\mu(W_i)}$$

log poles

$$\nabla_\mu = \nabla|_{\mathcal{V}_\mu} : g \cdot \prod_{i=1}^m z_i^{\mu(W_i)} \longmapsto \left(dg + g\left(\sum_{i=1}^m \mu(W_i) \frac{dz_i}{z_i} - \frac{n}{d} \cdot \frac{d(f_L \circ \pi)}{f_L \circ \pi} \right) \right) \cdot \prod_{i=1}^m z_i^{\mu(W_i)}$$

So $\nabla(\mathcal{V}_\mu) \subset \Omega^1_Y(\log E) \otimes \mathcal{V}_\mu$. And in particular, taking $y \in E_W$ to be generic,

we have $m = 1$, $E_1 = E_W$, and $\mathrm{Res}_{E_W} \nabla_\mu = \mu(W) - \frac{n}{d} \sum_{L_j \in W} a_j$ $\square$

Appendix 5: Indecomposability lemma

Lem: V linear space $\dim = n$. $A = \{L_1, \cdots, L_r\} \subset V$ indecomposable

$\Rightarrow \exists\ \varepsilon_1, \cdots, \varepsilon_r \in \mathbb{Q}$ s.t. $\sum_{L_j \in W} \varepsilon_j < \dim W$ for any nonzero proper subspace W.

Proof: Suppose $B = \{L_1, \cdots, L_n\} \subset A$ is a basis

So $\forall\ j > n$, $\exists\ \lambda_{j1}, \cdots, \lambda_{jn} \in \mathbb{C}$, $L_j = \lambda_{j1} L_1 + \cdots + \lambda_{jn} L_n$

Denote by $B_j = \{ L_i \mid 1 \le i \le n, \lambda_{ji} \neq 0 \}$, $\forall\ j > n$

$b_i = \# \{ j > n \mid L_i \in B_j \}$, $\forall\ 1 \le i \le n$

Take
$$\varepsilon_j = \begin{cases} \frac{n}{n+1}, & 1 \le j \le n \\ \frac{1}{n+1} \cdot \left(\sum_{L_i \in B_j} \frac{1}{b_i} \right), & j > n \end{cases}$$

Then
$$\sum_{j=1}^{r} \varepsilon_j = \frac{1}{n+1} \left(n^2 + \sum_{i=1}^{n} \frac{1}{b_i} \sum_{L_i \in B_j} 1 \right) = \frac{1}{n+1} \cdot (n^2 + n) = n$$

For any nonzero proper subspace $W \subsetneqq V$,

1) if $\dim W > \#(W \cap B)$,

then $\sum_{L_j \in W} \varepsilon_j \le \frac{n}{n+1} \cdot (\dim W - 1) + \frac{n}{n+1} < \dim W$

2) otherwise $\dim W = \#(W \cap B)$ i.e. $W = \mathrm{Span}(W \cap B)$.

Since A is indecomposable, $\exists\ j > n$ s.t. $B_j \cap W \neq \phi$, $B_j \not\subset W$

So
$$\sum_{L_j \in W} \varepsilon_j = \frac{n}{n+1} \cdot \dim W + \sum_{\substack{j > n \\ B_j \subset W}} \varepsilon_j$$

$$< \frac{n}{n+1} \cdot \dim W + \sum_{\substack{j > n \\ B_j \cap W \neq \phi}} \frac{1}{n+1} \cdot \sum_{L_i \in B_j} \frac{1}{b_i}$$

$$= \frac{n}{n+1} \cdot \dim W + \sum_{\substack{1 \le i \le n \\ L_i \in W}} \frac{1}{n+1} \cdot \frac{1}{b_i} \sum_{L_i \in B_j} 1 = \dim W \qquad \square$$